Lecture on Space Charge Effects*

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Space Charge Effects: Outline

1) Beam Space-Charge Model

2) Space Charge Effects in Transverse Beam Centroid and Envelope

3) Characteristic Transverse Particle Orbits Including Space-Charge

4) ~~Longitudinal Space-Charge Effects~~ (no time with 50 min)

5) Perspective

References

S1: Beam Space-Charge Model
Lorentz Force Equation

The *Lorentz force equation* of a charged particle is given by (MKS Units):

$$\boxed{\frac{d}{dt}\mathbf{p}_i(t) = q_i\left[\mathbf{E}(\mathbf{x}_i,t) + \mathbf{v}_i(t)\times\mathbf{B}(\mathbf{x}_i,t)\right]}$$

$m_i, \quad q_i$ .... particle mass, charge

$i =$ particle index

$t =$ time

$\mathbf{x}_i(t)$ .... particle coordinate

$\mathbf{p}_i(t) = m_i\gamma_i(t)\mathbf{v}_i(t)$ .... particle momentum

$\mathbf{v}_i(t) = \frac{d}{dt}\mathbf{x}_i(t) = c\vec{\beta}_i(t)$ .... particle velocity

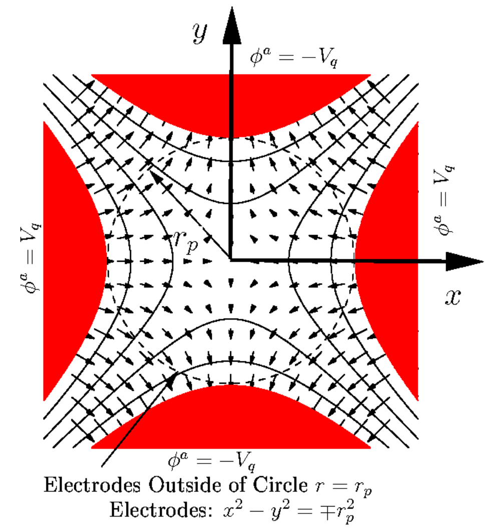
$\gamma_i(t) = \frac{1}{\sqrt{1-\beta_i^2(t)}}$ .... particle gamma factor

	Total		Applied		Self
Electric Field:	$\mathbf{E}(\mathbf{x},t)$	$=$	$\mathbf{E}^a(\mathbf{x},t)$	$+$	$\mathbf{E}^s(\mathbf{x},t)$
Magnetic Field:	$\mathbf{B}(\mathbf{x},t)$	$=$	$\mathbf{B}^a(\mathbf{x},t)$	$+$	$\mathbf{B}^s(\mathbf{x},t)$

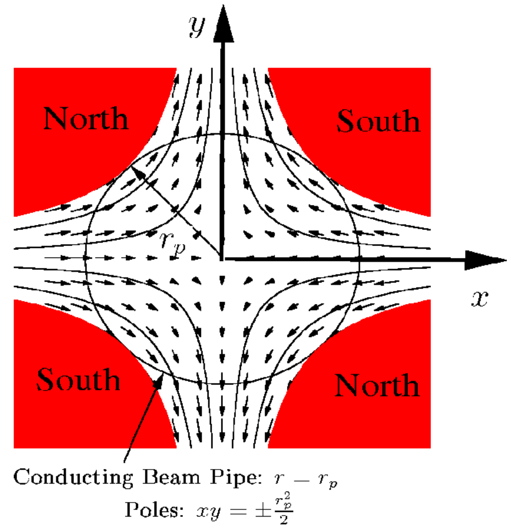
Applied Fields used to Focus, Bend, and Accelerate Beam

Transverse optics for focusing:

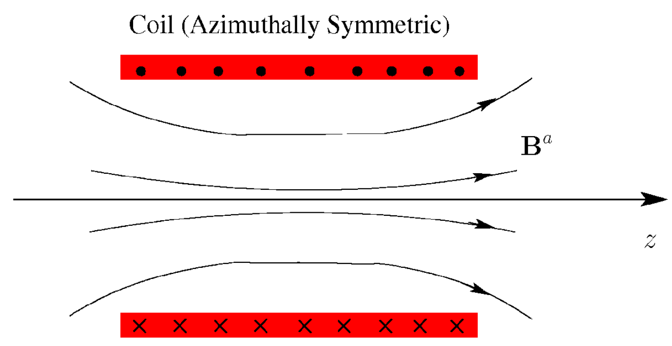
Electric Quadrupole



Magnetic Quadrupole

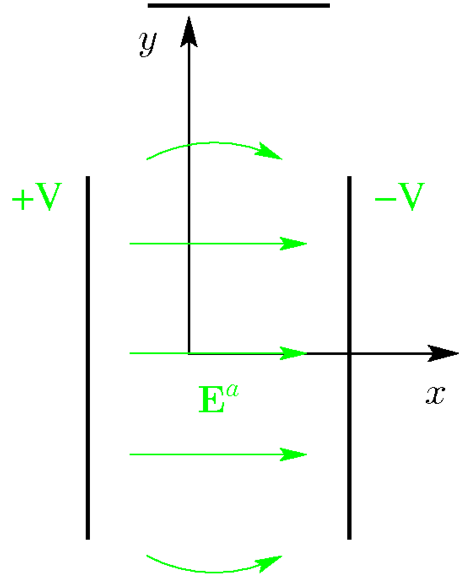


Solenoid

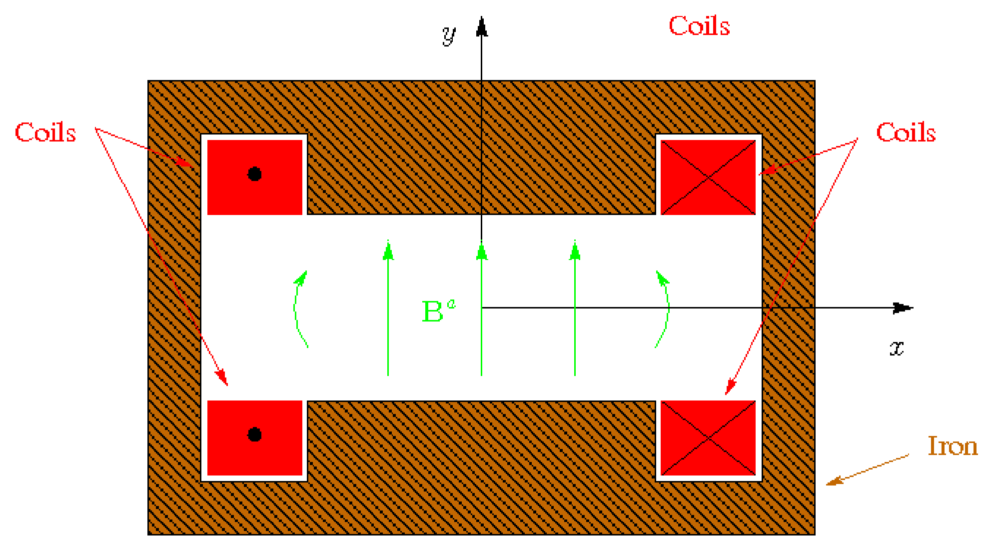


Dipole Bends:

Electric x-direction bend

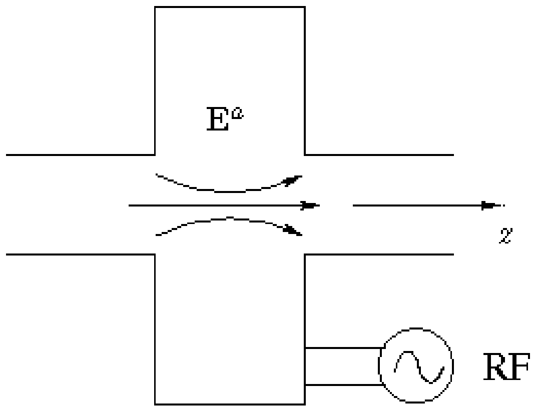


Magnetic x-direction bend



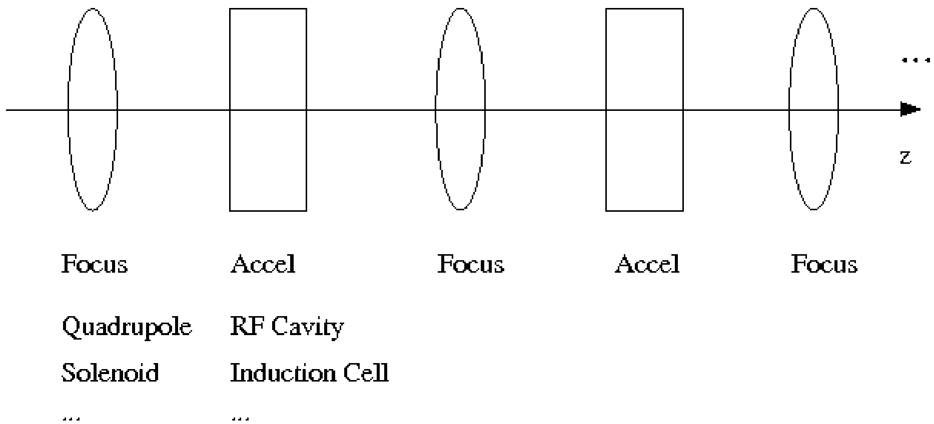
Longitudinal Acceleration:

RF Cavity



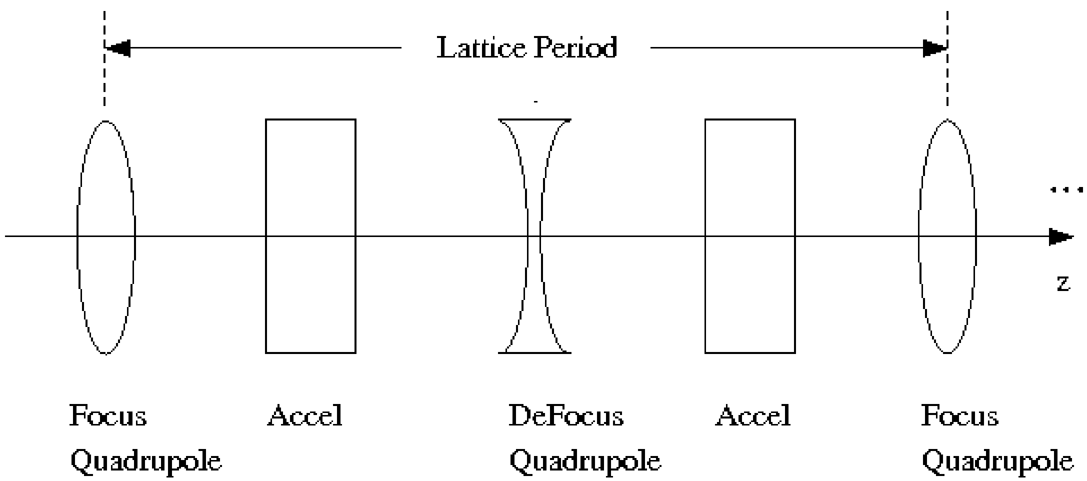
Machine Lattice

Applied field structures are often arraigned in a regular (periodic) lattice for beam transport/acceleration:

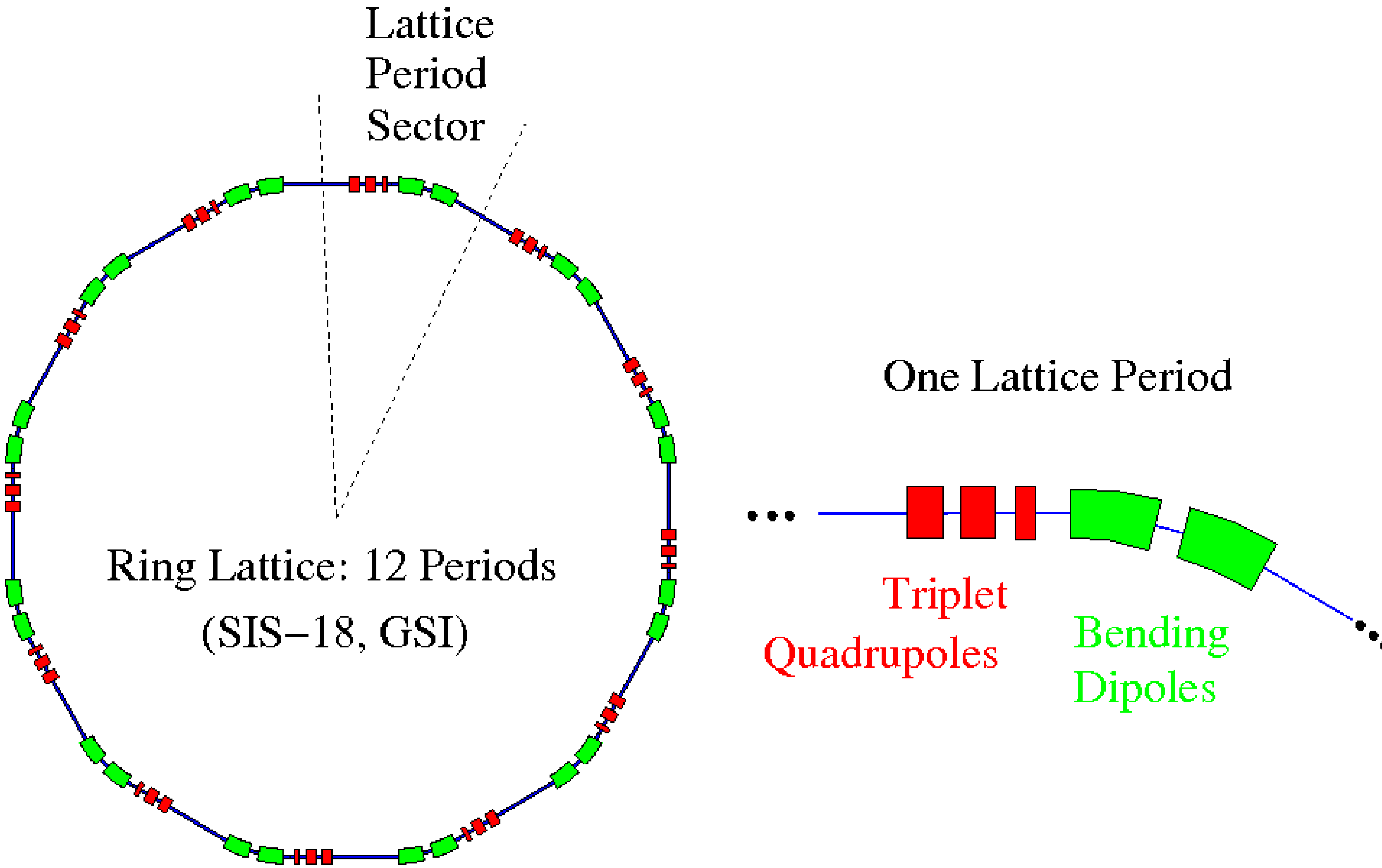


- Sometimes functions like bending/focusing are combined into a single element

Example – Linear FODO lattice (symmetric quadrupole doublet)



Lattices for rings and some beam insertion/extraction sections also incorporate bends and more complicated periodic structures:



- Elements to insert beam into and out of ring further complicate lattice
- Acceleration cells also present
  (typically several RF cavities at one or more location)

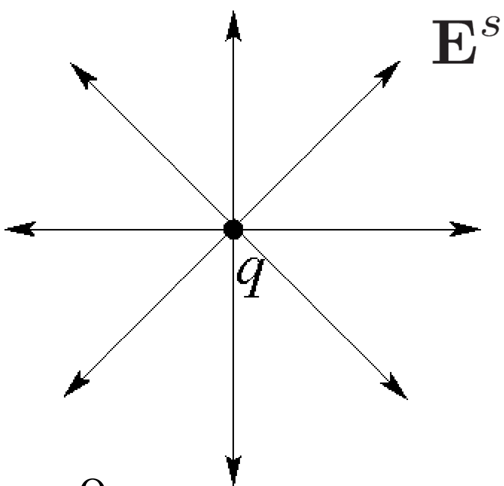
Beam Self fields

Self-fields are generated by the distribution of beam particles:

Charges + Currents

Particle at Rest

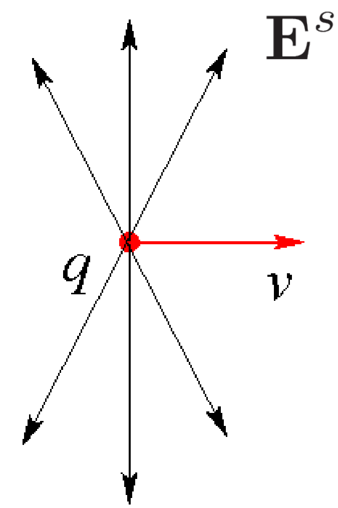
(pure electrostatic)

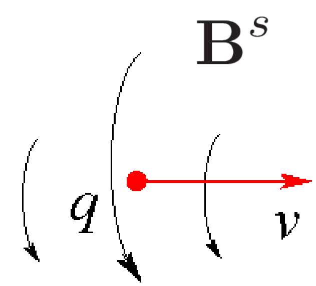


$\mathbf{B}^s = 0$

Particle in Motion

Obtain from Lorentz boost of rest-frame field: see Jackson, *Classical Electrodynamics*





- Superimpose for all particles in the beam distribution
- Accelerating particles also radiate
  - More important for electrons and light source applications. Neglect here.

The electric ($\mathbf{E}^a$) and magnetic ($\mathbf{B}^a$) fields satisfy the Maxwell Equations. The linear structure of the Maxwell equations can be exploited to resolve the field into Applied and Self-Field components:

$$\mathbf{E} = \mathbf{E}^a + \mathbf{E}^s$$

$$\mathbf{B} = \mathbf{B}^a + \mathbf{B}^s$$

Applied Fields (often quasi-static $\partial/\partial t \simeq 0$) $\mathbf{E}^a, \ \mathbf{B}^a$

Generated by elements in lattice

$$\nabla \cdot \mathbf{E}^a = \frac{\rho^a}{\epsilon_0} \qquad \nabla \times \mathbf{B}^a = \mu_0 \mathbf{J}^a$$

$$\nabla \times \mathbf{E}^a = 0 \qquad \nabla \cdot \mathbf{B}^a = 0$$

$$\rho^a = \text{applied charge density} \qquad \frac{1}{\mu_0 \epsilon_0} = c^2$$

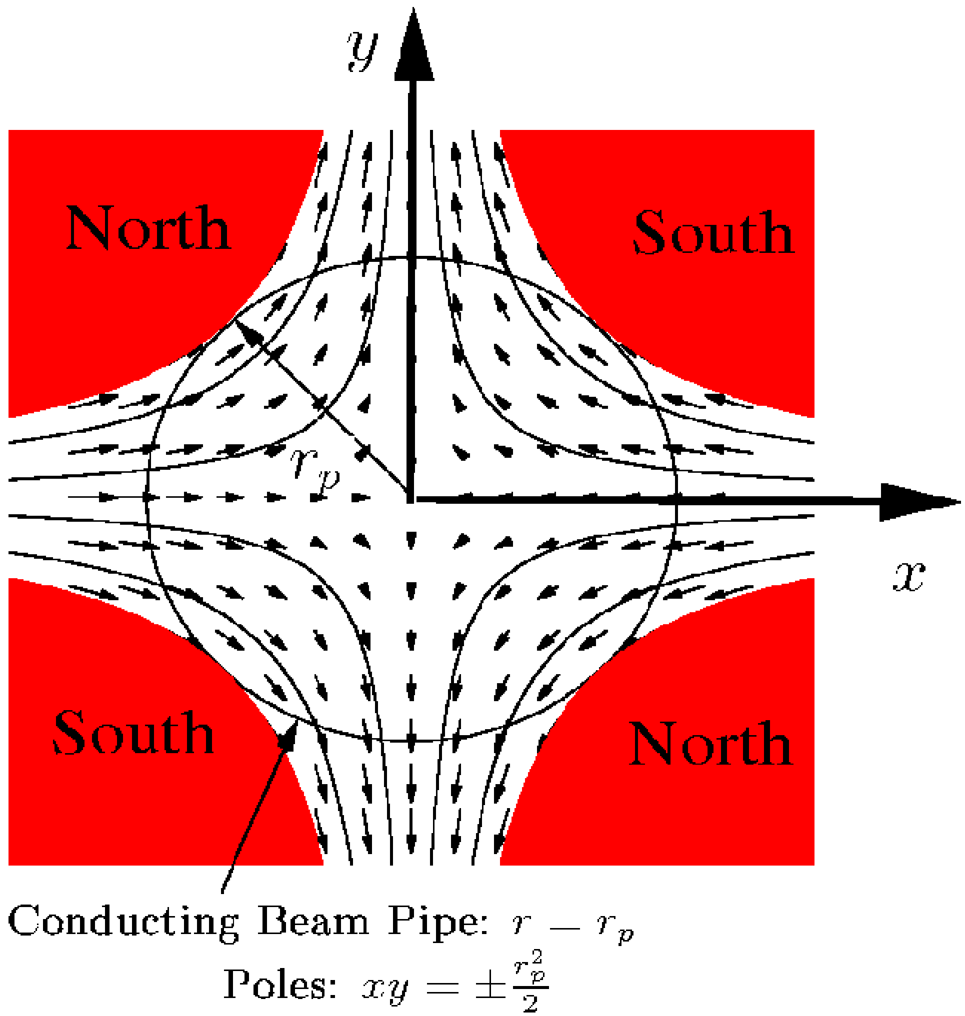
$$\mathbf{J}^a = \text{applied current density}$$

$$+ \text{ Boundary Conditions on } \mathbf{E}^a \text{ and } \mathbf{B}^a$$

- Boundary conditions depend on the total fields $\mathbf{E}, \ \mathbf{B}$ and if separated into Applied and Self-Field components, care needed
- System often a static boundary value problem and source free in the vacuum aperture of beam: $\nabla \cdot \mathbf{B}^a = 0 \qquad \nabla \times \mathbf{B}^a = 0$

Example Applied Field Element: 2D Magnetic Quadrupole

In the axial center of a long magnetic quadrupole, model fields as 2D transverse



- Magnetic (ideal iron) poles hyperbolic
- Structure infinitely extruded along z

2D Transverse Fields

$$\mathbf{E}^a = 0$$

$$B_x^a = Gy$$
$$B_y^a = Gx$$
$$B_z^a = 0$$

$$G \equiv \frac{B_q}{r_p} = \frac{\partial B_x^a}{\partial y} = \frac{\partial B_y^a}{\partial x}$$
$$= \text{Magnetic Gradient}$$

$$B_q = |\mathbf{B}^a|_{r=r_p} = \text{Pole Field}$$
$$r_p = \text{Pipe Radius}$$

Self-Fields (dynamic, evolve with beam)

Generated by particle of the beam rather than (applied) sources outside beam

$$\nabla \cdot \mathbf{E}^s = \frac{\rho^s}{\epsilon_0} \qquad \nabla \times \mathbf{B}^s = \mu_0 \mathbf{J}^s + \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}^s$$

$$\nabla \times \mathbf{E}^s = -\frac{\partial}{\partial t}\mathbf{B}^s \qquad \nabla \cdot \mathbf{B}^s = 0$$

$$\begin{aligned} \rho^s &= \text{beam charge density} \\ &= \sum_{i=1}^{N} q_i \delta[\mathbf{x} - \mathbf{x}_i(t)] \end{aligned}$$

$$\begin{aligned} \mathbf{J}^s &= \text{beam current density} \\ &= \sum_{i=1}^{N} q_i \mathbf{v}_i(t) \delta[\mathbf{x} - \mathbf{x}_i(t)] \end{aligned}$$

i = particle index (N particles)

q_i = particle charge

$\mathbf{x}_i$ = particle coordinate

$\mathbf{v}_i$ = particle velocity

$\delta(\mathbf{x}) \equiv \delta(x)\delta(y)\delta(z)$

$\delta(x) \equiv$ Dirac-delta function

$\sum_{i=1}^{N} \cdots =$ sum over beam particles

$+$ Boundary Conditions on $\mathbf{E}^s$ and $\mathbf{B}^s$ from material structures, radiation conditions, etc.

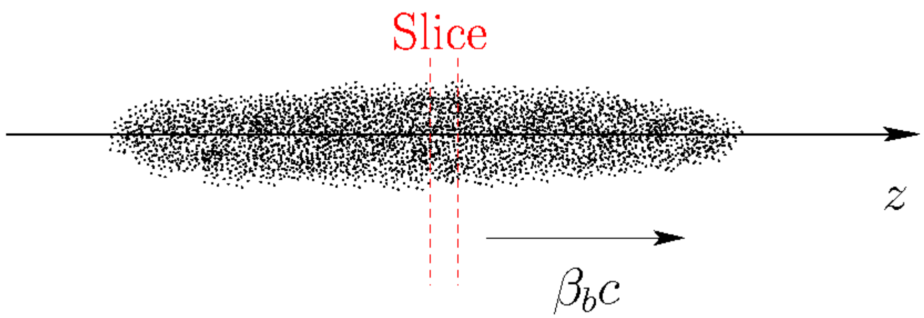
In accelerators, typically there is ideally a single species of particle:

$$\boxed{\begin{aligned} q_i &\rightarrow q \\ m_i &\rightarrow m \end{aligned}}$$

Large Simplification!

Multi-species results in more complex collective effects

Motion of particles within axial slices of the "bunch" are highly directed:



$$\beta_b(z)c \equiv \frac{1}{N'}\sum_{i=1}^{N'} \mathbf{v}_i \cdot \hat{\mathbf{z}}$$

$=$ Mean axial velocity of N' particles in beam slice

$$\frac{d}{dt}\mathbf{x}_i(t) = \mathbf{v}_i(t) = \hat{\mathbf{z}}\beta_b(z)c + \delta\mathbf{v}_i$$

$$|\delta\mathbf{v}_i| \ll |\beta_b|c$$

Paraxial Approximation

There are typically many particles: so apply a continuum approximation

$$\rho^s = \sum_{i=1}^{N} q_i \delta[\mathbf{x} - \mathbf{x}_i(t)] \simeq \rho(\mathbf{x}, t)$$ continuous charge-density

$$\mathbf{J}^s = \sum_{i=1}^{N} q_i \mathbf{v}_i(t)\delta[\mathbf{x} - \mathbf{x}_i(t)] \simeq \beta_b c \rho(\mathbf{x}, t)\hat{\mathbf{z}}$$ continuous axial current-density

The beam evolution is typically sufficiently slow where we can neglect radiation and approximate the self-field Maxwell Equations as:

See: Appendix A, Magnetic Self-Fields and

$$\boxed{\begin{aligned} &\mathbf{E}^s = -\nabla\phi \\ &\mathbf{B}^s = \nabla \times \mathbf{A} \qquad\qquad \mathbf{A} = \hat{\mathbf{z}}\frac{\beta_b}{c}\phi \\ &\nabla^2\phi = \frac{\partial}{\partial\mathbf{x}}\cdot\frac{\partial}{\partial\mathbf{x}}\phi = -\frac{\rho^s}{\epsilon_0} \\ &+\ \text{Boundary Conditions on } \phi \end{aligned}}$$

Vast Reduction of self-field model:

Approximation equiv to electrostatic interactions in frame moving with beam: see Appendix A

But still complicated

Resolve the Lorentz force acting on beam particles into Applied and Self-Field terms:

$$\mathbf{F}_i(\mathbf{x}_i, t) = q\mathbf{E}(\mathbf{x}_i, t) + q\mathbf{v}_i(t) \times \mathbf{B}(\mathbf{x}_i, t)$$

$$\mathbf{F}_i = \mathbf{F}_i^a + \mathbf{F}_i^s$$
$$\mathbf{E} = \mathbf{E}^a + \mathbf{E}^s$$
$$\mathbf{B} = \mathbf{B}^a + \mathbf{B}^s$$

Applied:

$$\boxed{\mathbf{F}_i^a = q\mathbf{E}_i^a + q\mathbf{v}_i \times \mathbf{B}_i^a}$$

$$\mathbf{E}^a(\mathbf{x}_i, t) \equiv \mathbf{E}_i^a \quad \text{etc.}$$

Self-Field:

$$\boxed{\mathbf{F}_i^s = q\mathbf{E}_i^s + q\mathbf{v}_i \times \mathbf{B}_i^s}$$

The self-field force can be simplified:

Plug in self-field forms:

~0 Neglect: Paraxial

$$\cdots\Big|_i \equiv \cdots\Big|_{\mathbf{x}=\mathbf{x}_i}$$

$$
\begin{aligned}
\mathbf{F}_i^s &= q\mathbf{E}_i^s + q\mathbf{v}_i \times \mathbf{B}_i^s \\
&\simeq q\left[-\frac{\partial \phi}{\partial \mathbf{x}}\Big|_i + (\beta_b c \hat{\mathbf{z}} + \delta \mathbf{v}_i) \times \left(\frac{\partial}{\partial \mathbf{x}} \times \hat{\mathbf{z}} \frac{\beta_b}{c} \phi \right)\Big|_i \right]
\end{aligned}
$$

Resolve into transverse (x and y) and longitudinal (z) components and simplify:

$$
\begin{aligned}
\beta_b c \hat{\mathbf{z}} \times \left(\frac{\partial}{\partial \mathbf{x}} \times \hat{\mathbf{z}} \frac{\beta_b}{c} \phi \right)\Big|_i &= \beta_b^2 \hat{\mathbf{z}} \times \left(\frac{\partial}{\partial \mathbf{x}_\perp} \times \hat{\mathbf{z}} \phi \right)\Big|_i \\
&= \beta_b^2 \hat{\mathbf{z}} \times \left(\frac{\partial \phi}{\partial y} \hat{\mathbf{x}} - \frac{\partial \phi}{\partial x} \hat{\mathbf{y}} \right)\Big|_i \\
&= \beta_b^2 \left(\frac{\partial \phi}{\partial x} \hat{\mathbf{x}} + \frac{\partial \phi}{\partial y} \hat{\mathbf{y}} \right)\Big|_i \\
&= \beta_b^2 \frac{\partial \phi}{\partial \mathbf{x}_\perp}\Big|_i
\end{aligned}
$$

also

$$-\left.\frac{\partial \phi}{\partial \mathbf{x}}\right|_i = -\left.\frac{\partial \phi}{\partial \mathbf{x}_\perp}\right|_i - \left.\frac{\partial \phi}{\partial \mathbf{z}}\right|_i \hat{\mathbf{z}}$$

Together, these results give:

$$\boxed{\mathbf{F}_i^s = \underbrace{\boxed{-\frac{q}{\gamma_b^2}\left.\frac{\partial \phi}{\partial \mathbf{x}_\perp}\right|_i}}_{\text{Transverse}} \underbrace{\boxed{-\hat{\mathbf{z}}q\left.\frac{\partial \phi}{\partial z}\right|_i}}_{\text{Longitudinal}}}$$

$$\gamma_b \equiv \frac{1}{\sqrt{1-\beta_b^2}}$$ Axial relativistic gamma of beam

- Transverse and longitudinal forces have different axial gamma factors
- $1/\gamma_b^2$ factor in transverse force shows the space-charge forces become weaker as axial beam kinetic energy increases
  - Most important in low energy (nonrelativistic) beam transport
  - Strong in/near injectors before much acceleration

The particle equations of motion in $\mathbf{x}_i - \mathbf{v}_i$ phase-space variables become:

- Separate parts of $q\mathbf{E}^a_i + q\mathbf{v}_i \times \mathbf{B}^a_i$ into transverse and longitudinal comp

Transverse

$$\frac{d}{dt}\mathbf{x}_{\perp i} = \mathbf{v}_{\perp i}$$

$$\frac{d}{dt}(m\gamma_i \mathbf{v}_{\perp i}) \simeq \underbrace{\boxed{q\mathbf{E}^a_{\perp i} + q\beta_b c\hat{\mathbf{z}} \times \mathbf{B}^a_{\perp i} + qB^a_{zi}\mathbf{v}_{\perp i} \times \hat{\mathbf{z}}}}_{\text{Applied}} \underbrace{\boxed{- q\frac{1}{\gamma_b^2}\frac{\partial \phi}{\partial \mathbf{x}_\perp}\bigg|_i}}_{\text{Self}}$$

Longitudinal

$$\frac{d}{dt}z_i = v_{zi}$$

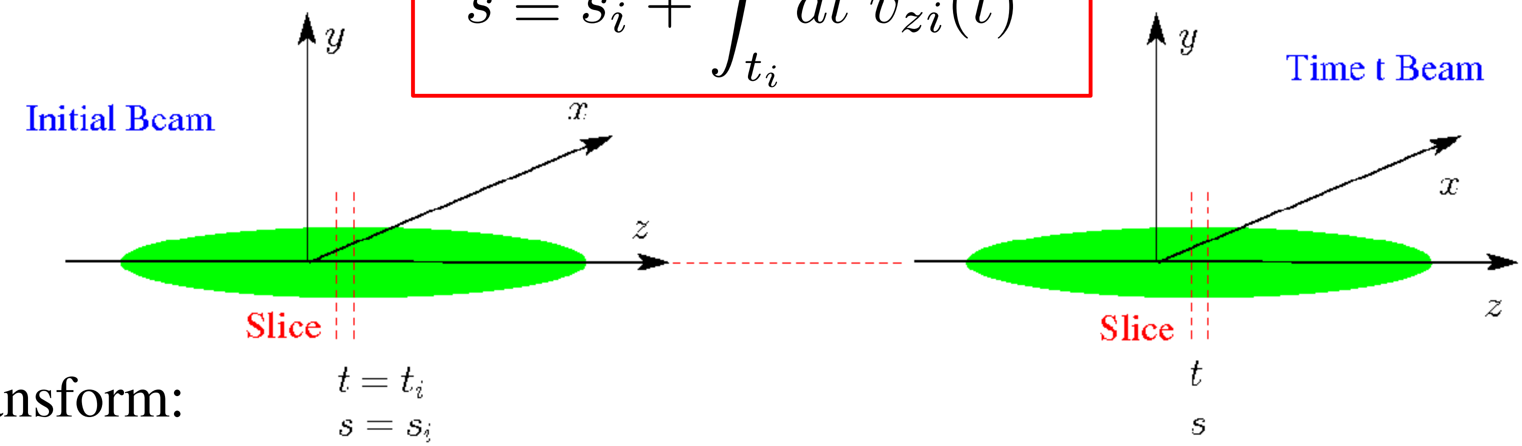
$$\frac{d}{dt}(m\gamma_i v_{zi}) \simeq \underbrace{\boxed{qE^a_{zi} - q(v_{xi}B^a_{yi} - v_{yi}B^a_{xi})}}_{\text{Applied}} \underbrace{\boxed{- q\frac{\partial \phi}{\partial z}\bigg|_i}}_{\text{Self}}$$

Equations of Motion in s and the Paraxial Approximation

In transverse accelerator dynamics, it is convenient to employ the axial coordinate (s) of a particle in the accelerator as the independent variable:

- Need fields at lattice location of particle to integrate equations for particle trajectories

$$\boxed{s \equiv s_i + \int_{t_i}^{t} d\tilde{t}\, v_{zi}(\tilde{t})}$$



Transform:

Neglect

$$v_{zi} = \frac{ds}{dt} \quad \Longrightarrow \quad v_{xi} = \frac{dx_i}{dt} = \frac{ds}{dt}\frac{dx_i}{ds} = v_{zi}\frac{dx_i}{ds} = (\beta_b c + \delta v_{zi})\frac{dx_i}{ds}$$

$$\simeq \beta_b c \frac{dx_i}{ds}$$

Denote:

$$\boxed{{}' \equiv \frac{d}{ds} \qquad \begin{aligned} v_{xi} &= \frac{dx_i}{dt} \simeq \beta_b c x_i' \\ v_{yi} &= \frac{dy_i}{dt} \simeq \beta_b c y_i' \end{aligned}}$$

Neglecting term consistent with assumption of small longitudinal momentum spread (paraxial approximation)

- Procedure becomes more complicated when bends present

In the paraxial approximation, x' and y' can be interpreted as the (small magnitude) angles that the particles make with the longitudinal-axis:

$$\boxed{\begin{aligned} x-\text{angle} &= \frac{v_{xi}}{v_{zi}} \simeq \frac{v_{xi}}{\beta_b c} = x'_i \\ y-\text{angle} &= \frac{v_{yi}}{v_{zi}} \simeq \frac{v_{yi}}{\beta_b c} = y'_i \end{aligned}}$$

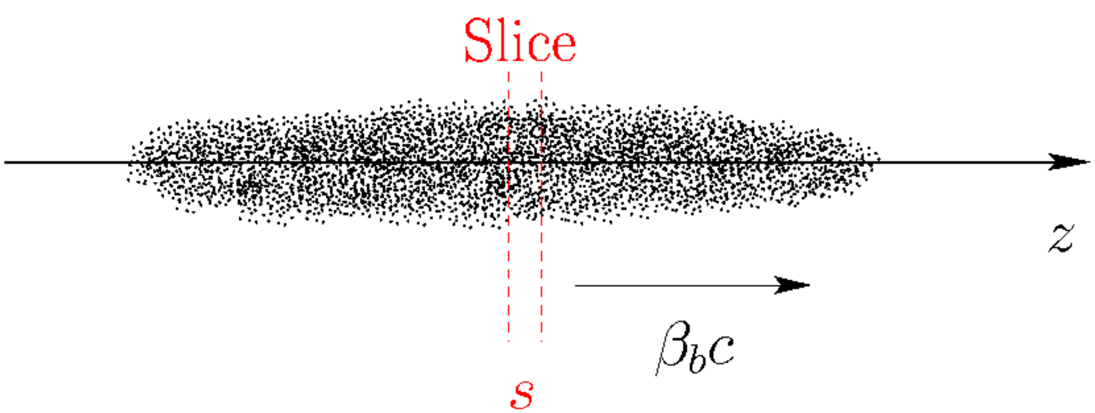
Typical accel lattice values:

$|x'| < 50$ mrad

The angles will be *small* in the paraxial approximation:

$$v_{xi}^2,\ v_{yi}^2 \ll \beta_b^2 c^2 \quad \Longrightarrow \quad x_i'^2,\ y_i'^2 \ll 1$$

Since the spread of axial momentum/velocities is small in the paraxial approximation, a thin axial slice of the beam maps to a thin axial slice and s can also be thought of as the axial coordinate of the slice in the accelerator lattice



$$\beta_b \equiv \sum_{i=1}^{N'} \frac{v_{zi}}{c}$$

slice

$$\boxed{s \simeq s_i + c\int_{t_i}^{t} d\tilde{t}\ \beta_b(\tilde{t})}$$

$$\boxed{s \simeq s_i + c \int_{t_i}^{t} d\tilde{t}\; \beta_b(\tilde{t})}$$

The coordinate *s* can alternatively be interpreted as the axial coordinate of a reference (design) particle moving in the lattice

- Design particle has no momentum spread

It is often desirable to express the particle equations of motion in terms of *s* rather than the time *t*

- Makes it clear where you are in the lattice of the machine
- Sometimes easier to use t in codes when including many effects to high order

Transform transverse particle equations of motion to *s* rather than *t* derivatives

$$\boxed{\frac{d}{dt}(m\gamma_i \mathbf{v}_{\perp i})} \simeq q\mathbf{E}^a_{\perp i} + q\beta_b c\hat{\mathbf{z}} \times \mathbf{B}^a_{\perp i} + \boxed{qB^a_{zi}\mathbf{v}_{\perp i} \times \hat{\mathbf{z}}} - q\frac{1}{\gamma_b^2}\left.\frac{\partial\phi}{\mathbf{x}_\perp}\right|_i$$

Term 1 Term 2

Transform Terms 1 and 2 in the particle equation of motion:

$$\frac{d}{dt} = v_{zi}\frac{d}{ds}$$

Term 1: $$\frac{d}{dt}\left(m\gamma_i\frac{d\mathbf{x}_{\perp i}}{dt}\right) = mv_{zi}\frac{d}{ds}\left(\gamma_i v_{zi}\frac{d}{ds}\mathbf{x}_{\perp i}\right)$$

$$= \underbrace{m\gamma_i v_{zi}^2\frac{d^2}{ds^2}\mathbf{x}_{\perp i}}_{\text{Term 1A}} + \underbrace{mv_{zi}\left(\frac{d}{ds}\mathbf{x}_{\perp i}\right)\frac{d}{ds}(\gamma_i v_{zi})}_{\text{Term 1B}}$$

Approximate:

Term 1A: $$m\gamma_i v_{zi}^2\frac{d^2}{ds^2}\mathbf{x}_{\perp i} \simeq m\gamma_b\beta_b^2c^2\frac{d^2}{ds^2}\mathbf{x}_{\perp i} = m\gamma_b\beta_b^2c^2\mathbf{x}''_{\perp i}$$

Term 1B: $$mv_{zi}\left(\frac{d}{ds}\mathbf{x}_{\perp i}\right)\frac{d}{ds}(\gamma_i v_{zi}) \simeq m\beta_b c\left(\frac{d}{ds}\mathbf{x}_{\perp i}\right)\frac{d}{ds}(\gamma_b\beta_b c)$$

$$\simeq m\beta_b c^2(\gamma_b\beta_b)'\mathbf{x}'_{\perp i}$$

Using the approximations 1A and 1B gives for Term 1:

$$\boxed{m\frac{d}{dt}\left(\gamma_i \frac{d\mathbf{x}_{\perp i}}{dt}\right) \simeq m\gamma_b\beta_b^2 c^2 \left[\mathbf{x}''_{\perp i} + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\mathbf{x}'_{\perp i}\right]}$$

Similarly we approximate in Term 2:

$$\boxed{qB^a_{zi}\mathbf{v}_{\perp i} \times \hat{\mathbf{z}} \simeq qB^a_{zi}\beta_b c\mathbf{x}'_{\perp i} \times \hat{\mathbf{z}}}$$

Using the simplified expressions for Terms 1 and 2 obtain the reduced transverse equation of motion:

$$\boxed{\begin{aligned}\mathbf{x}''_{\perp i} + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\mathbf{x}'_{\perp i} = & \frac{q}{m\gamma_b\beta_b^2 c^2}\mathbf{E}^a_{\perp i} + \frac{q}{m\gamma_b\beta_b c}\hat{\mathbf{z}} \times \mathbf{B}^a_{\perp i} \\ & + \frac{qB^a_{zi}}{m\gamma_b\beta_b c}\mathbf{x}'_{\perp i} \times \hat{\mathbf{z}} - \frac{q}{m\gamma_b^3\beta_b^2 c^2}\left.\frac{\partial \phi}{\partial \mathbf{x}_\perp}\right|_i\end{aligned}}$$

Summary: Transverse Particle Equations of Motion

$$\mathbf{x}_\perp'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\mathbf{x}_\perp' = \frac{q}{m\gamma_b\beta_b^2c^2}\mathbf{E}_\perp^a + \frac{q}{m\gamma_b\beta_b c}\hat{\mathbf{z}}\times\mathbf{B}_\perp^a + \frac{qB_z^a}{m\gamma_b\beta_b c}\mathbf{x}_\perp'\times\hat{\mathbf{z}}$$

$$-\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial\mathbf{x}_\perp}\phi$$

$$\mathbf{E}^a = \text{Applied Electric Field} \qquad {}' \equiv \frac{d}{ds} \qquad \gamma_b \equiv \frac{1}{\sqrt{1-\beta_b^2}}$$

$$\mathbf{B}^a = \text{Applied Magnetic Field}$$

$$\nabla^2\phi = \frac{\partial}{\partial\mathbf{x}}\cdot\frac{\partial}{\partial\mathbf{x}}\phi = -\frac{\rho}{\epsilon_0}$$

$$+ \text{ Boundary Conditions on } \phi$$

Drop particle *i* subscripts (in most cases) henceforth to simplify notation

Neglects axial energy spread, bending, and electromagnetic radiation

$\gamma-$ factors different in applied and self-field terms:

In $\displaystyle -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial\mathbf{x}}\phi,$ contributions to γ_b^3 :

$\gamma_b \Longrightarrow$ Kinematics

$\gamma_b^2 \Longrightarrow$ Self-Magnetic Field Corrections (leading order)

Summary of Transverse Particle Equations of Motion

In a quadrupole magnetic focusing channel, without momentum spread, bends, radiation, the particle equations of motion in both the *x*- and *y*-planes expressed as:

Accel Applied Space-Charge

$$\boxed{\begin{aligned} x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x(s)x &= -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial x}\phi \\ y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' + \kappa_y(s)y &= -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial y}\phi \end{aligned}}$$

Quadrupole Focusing Lattice:

$$\mathrm{B}_x^a = Gy \qquad B_y^a = Gx$$ **Field**

$$\kappa_x(s) = -\kappa_y(s) = \frac{G(s)}{[B\rho]}$$

$$G(s) = \frac{\partial B_x^a}{\partial y} = \frac{\partial B_y^a}{\partial x} = \frac{B_p}{r_p}$$ **Gradient**

$$[B\rho] = \frac{m\gamma_b\beta_b c}{q}$$ **Rigidity**

In more advanced treatments cases can be put into this same basic form. This will be covered in graduate **Accelerator Physics**

- Electric Quadrupoles (rescale $\kappa_x, \ \kappa_y$), Solenoids (frame transform)
- Weak Acceleration (normalized variables) $(\gamma_b\beta_b) \simeq \text{const}$

Reminder: Hill's Equation for Yue Hao lectures

Neglect:

- Space-charge effects: $\partial\phi/\partial\mathbf{x} \simeq 0$
- Nonlinear applied focusing and bends: $\mathbf{B}^a$ has only linear focus terms
- Acceleration: $\gamma_b\beta_b \simeq \text{const}$
- Momentum spread effects: $v_{zi} \simeq \beta_b c$

Then the transverse particle equations of motion reduce to Hill's Equation:

$$\boxed{x''(s) + \kappa(s)x(s) = 0}$$

x=⊥ particle coordinate

(i.e., x or y or possibly combinations of coordinates)

$s =$ Axial coordinate of reference particle

$$\prime = \frac{d}{ds}$$

$\kappa(s) =$ Lattice focusing function (linear fields)

Hao lectures covered transfer matrices, stability, phase-amplitude methods, phase-space area, etc. with Hill's equation. How does space-charge change?

Appendix A: Magnetic Self-Fields

The full Maxwell equations for the beam self fields

$$\mathbf{E}^s, \quad \mathbf{B}^s$$

with electromagnetic effects neglected can be written as

- Good approx typically for slowly varying ions in weak fields

$$\nabla \cdot \mathbf{E}^s = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{B}^s = \mu_0 \mathbf{J} + \frac{1}{c^2} \cancel{\frac{\partial}{\partial t}} \mathbf{E}^s$$

$$\nabla \times \mathbf{E}^s = -\cancel{\frac{\partial}{\partial t}} \mathbf{B}^s \qquad \nabla \cdot \mathbf{B}^s = 0$$

$$+ \text{ Boundary Conditions on } \mathbf{E}^s \text{ and } \mathbf{B}^s$$
$$. \qquad \text{from material structures, etc.}$$

$$\rho = qn(\mathbf{x}, t) \qquad n(\mathbf{x}, t) = \text{Number Density}$$

$$\mathbf{J} = qn(\mathbf{x}, t)\mathbf{V}(\mathbf{x}, t) \qquad \mathbf{V}(\mathbf{x}, t) = \text{"Fluid" Flow Velocity}$$

- Beam terms from charged particles making up the beam
- Calc from continuum approx distribution

Electrostatic Approx:

$$\boxed{\begin{aligned}\nabla \cdot \mathbf{E}^s &= \frac{qn}{\epsilon_0} \\ \nabla \times \mathbf{E}^s &= 0\end{aligned}}$$

$$\mathbf{E}^s = -\nabla\phi$$

$$\phi = \text{Electrostatic Scalar Potential}$$

$$\implies \quad \nabla \times \mathbf{E}^s = -\nabla \times \nabla\phi = 0$$

Continuity of mixed partial derivatives

$$\implies \quad \nabla \cdot \mathbf{E}^s = -\nabla \cdot \nabla\phi = \frac{qn}{\epsilon_0}$$

$$\boxed{\begin{aligned}&\nabla^2\phi = -\frac{qn}{\epsilon_0} \\ &+ \text{Boundary Conditions on } \phi\end{aligned}}$$

Magnetostatic Approx:

$$\boxed{\begin{aligned}\nabla \times \mathbf{B}^s &= \mu_0\mathbf{J} \\ \nabla \cdot \mathbf{B}^s &= 0\end{aligned}}$$

$$\mathbf{B}^s = \nabla \times \mathbf{A}$$

$$\mathbf{A} = \text{Magnetostatic Vector Potential}$$

$$\implies \nabla \cdot \mathbf{B}^s = \nabla \cdot (\nabla \times \mathbf{A}) = 0$$

Continuity of mixed partial derivatives

$$\implies \nabla \times \mathbf{B}^s = \nabla \times (\nabla \times \mathbf{A}) = \mu_0\mathbf{J}$$

Continue next slide

Magnetostatic Approx Continued:

$$\nabla \times \mathbf{B}^s = \nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J}$$

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$$

Still free to take (gauge choice):

$$\nabla \cdot \mathbf{A} = 0 \quad \text{Coulomb Gauge}$$

Can always meet this choice:

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla\xi \qquad \xi = \text{Some Function}$$

0 Cont mixed partial derivatives

$$\Longrightarrow \quad \mathbf{B}^s = \nabla \times \mathbf{A} \rightarrow \nabla \times \mathbf{A} + \nabla \times \nabla\xi = \nabla \times \mathbf{A}$$

$$\Longrightarrow \nabla \cdot \mathbf{A} \rightarrow \nabla \cdot \mathbf{A} + \nabla^2 \xi$$

Can always choose ξ such that $\nabla \cdot \mathbf{A} = 0$ to satisfy the Coulomb gauge:

$$\boxed{\begin{aligned} &\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} = -\mu_0 q n \mathbf{V} \\ &+ \text{Boundary Conditions on } \mathbf{A} \end{aligned}}$$

- Essentially one Poisson form eqn for each field *x*,*y*,*z* comp
- Boundary conditions diff than ϕ

But can approximate this further for “typical” paraxial beams …..

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} = -\mu_0 q n \mathbf{V}$$

Expect for a beam with primarily forward (paraxial) directed motion:

$$V_z = \beta_b c \qquad V_{x,y} \sim R' \beta_b c \qquad R' = \text{Beam Envelope Angle}$$
$$\text{(Typically 10s mrad Magnitude)}$$

$$\Longrightarrow \quad |A_{x,y}| \ll |A_z|$$

Giving:

$$\nabla^2 A_z = -\mu_0 q \beta_b c n \qquad n = -\frac{\epsilon_0}{q} \nabla^2 \phi$$

Free to use from electrostatic part

$$\nabla^2 A_z = (\mu_0 \epsilon_0) c \beta_b \nabla^2 \phi \qquad \mu_0 \epsilon_0 = \frac{1}{c^2}$$

From unit definition

$$\nabla^2 A_z = \frac{\beta_b}{c} \nabla^2 \phi$$

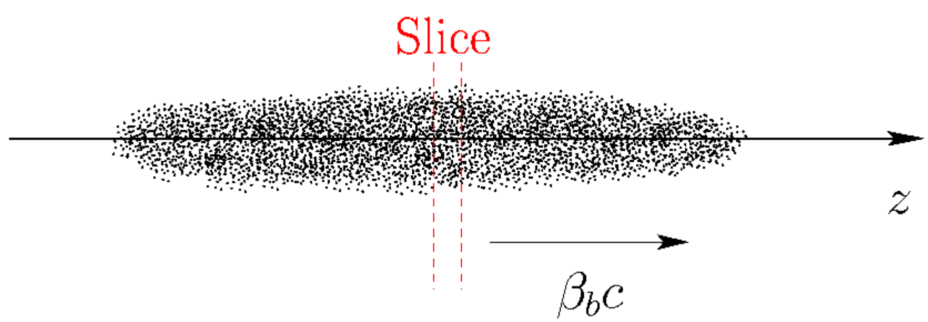
$$\Longrightarrow \boxed{A_z = \frac{\beta_b}{c} \phi}$$

- Allows simply taking into account low-order self-magnetic field effects
  - Care must be taken if magnetic materials are present close to beam

Further insight can be obtained on the nature of the approximations in the reduced form of the self-magnetic field correction by examining
Lorentz Transformation properties of the potentials.

From EM theory, the potentials $\phi,\ c\mathbf{A}$ form a relativistic 4-vector that transforms as a Lorentz vector for covariance:

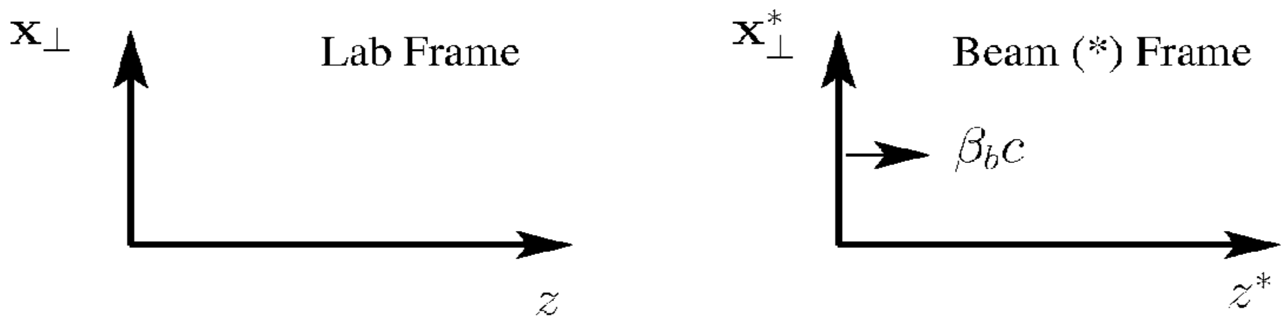
$$A_\mu = (\phi,\ c\mathbf{A})$$



In the rest frame (*) of the beam, assume that the flows are small enough where the potentials are purely electrostatic with:

$$A^*_\mu = (\phi^*, \mathbf{0}) \qquad \nabla^2 \phi^* = -\frac{qn^*}{\epsilon_0}$$

Review: Under Lorentz transform, the 4-vector components of $A_\mu = (\phi, c\mathbf{A})$ transform as the familiar 4-vector $x_\mu = (ct, \mathbf{x})$



<u>Transform</u>

$$\begin{aligned} ct^* &= \gamma_b(ct - \beta_b z) \\ z^* &= \gamma_b(z - \beta_b ct) \\ \mathbf{x}^* &= \mathbf{x}_\perp \end{aligned}$$

<u>Inverse Transform</u>

$$\begin{aligned} ct &= \gamma_b(ct^* + \beta_b z^*) \\ z &= \gamma_b(z^* + \beta_b ct^*) \\ \mathbf{x} &= \mathbf{x}_\perp^* \end{aligned}$$

This gives for the 4-potential $A_\mu = (\phi, c\mathbf{A})$:

$$\phi = \gamma_b(\phi^* + \beta_b c\overset{0}{A_z^*}) = \gamma_b\phi^*$$

$$cA_z = \gamma_b(c\overset{0}{A_z^*} + \beta_b\phi^*) = \beta_b(\gamma_b\phi^*) = \beta_b\phi$$

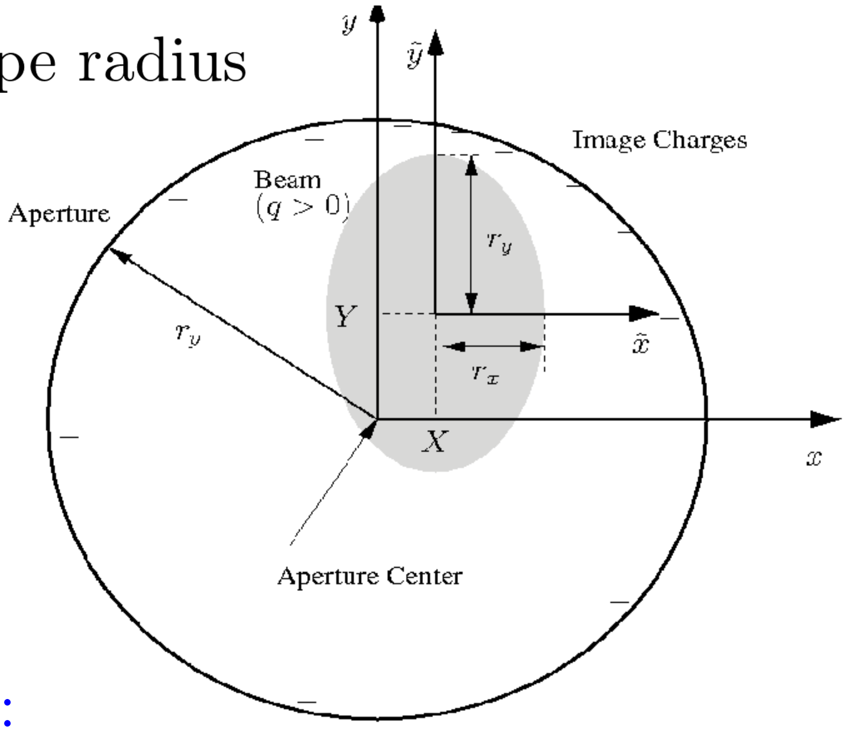
$$\Longrightarrow \boxed{A_z = \frac{\beta_b}{c}\phi}$$

- Shows result is consistent with pure electrostatic in beam (*) frame

S2: Space-Charge Effects: Transverse Centroid and Envelope

Analyze transverse centroid and envelope properties of an unbunched $(\partial/\partial z = 0)$ beam

r_p = pipe radius



Expect for linearly focused beam with intense space-charge:

- Beam to look roughly elliptical in shape
- Nearly uniform density within fairly sharp edge

Transverse averages:

$$\langle \cdots \rangle_\perp \equiv \frac{\int d^2x_\perp \int d^2x'_\perp \; \cdots f_\perp}{\int d^2x_\perp \int d^2x'_\perp \; f_\perp}$$

Centroid:

$$\boxed{\begin{aligned} X &= \langle x \rangle_\perp \\ Y &= \langle y \rangle_\perp \end{aligned}}$$

x- and *y*-coordinates of beam "center of mass"

Envelope: (edge measure)

$$\boxed{\begin{aligned} r_x &= 2\sqrt{\langle (x-X)^2 \rangle_\perp} \\ r_y &= 2\sqrt{\langle (y-Y)^2 \rangle_\perp} \end{aligned}}$$

x- and *y*-principal axis radii of an elliptical beam envelope

- Apply to general $f_\perp$ but base on uniform density $f_\perp$
- Factor of 2 results from dimensionality (diff 1D and 3D)

Oscillations in the statistical beam centroid and envelope radii are the *lowest-order* collective responses of the beam

Centroid Oscillations: Associated with errors and are suppressed to the extent possible:

- Error Sources seeding/driving oscillations:
  - Beam distribution assymetries (even emerging from injector: born offset)
  - Dipole bending terms from imperfect applied field optics
  - Dipole bending terms from imperfect mechanical alignment
- Exception: Large centroid oscillations desired when the beam is kicked (insertion or extraction) into or out of a transport channel as is done in beam insertion/extraction in/out of rings

Envelope Oscillations: Can have two components in periodic focusing lattices

1) Matched Envelope: Periodic "flutter" synchronized to period of focusing lattice to maintain best radial confinement of the beam

- Properly tuned flutter essential in Alternating Gradient quadrupole lattices

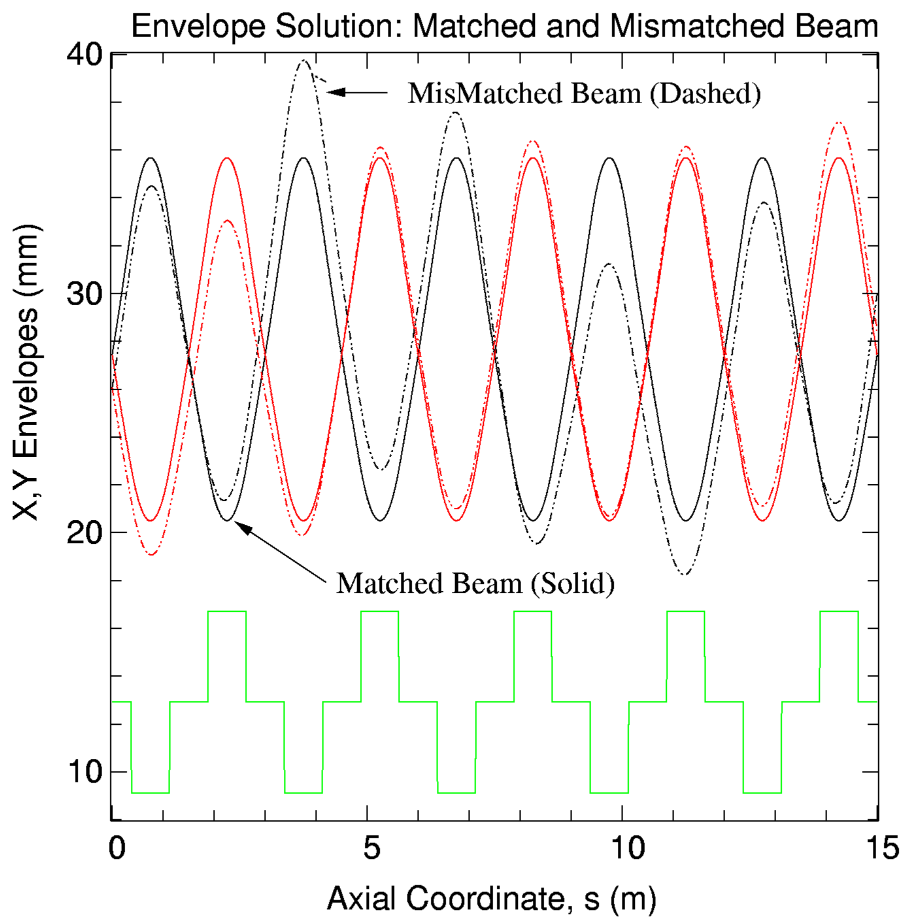
2) Mismatched Envelope: Excursions deviate from matched flutter motion and are seeded/driven by errors

Limiting maximum beam-edge excursions is desired for economical transport

- Reduces cost by Limiting material volume needed to transport an intense beam
- Reduces generation of halo (particles outside distribution core) and particle loses

Mismatched beams have larger envelope excursions and have more collective stability and beam halo problems since mismatch adds another source of free energy that can drive statistical increases in particle amplitudes

Example: FODO Quadrupole Transport Channel



Black: x–envelope
Red: y–envelope
Green: x–focusing

- Larger machine aperture is needed to confine a mismatched beam
  - Even in absence of beam halo and other mismatch driven "instabilities"

Centroid and Envelope oscillations are the *most important collective modes* of an intense beam

- Force balances based on matched beam envelope equation predict scaling of transportable beam parameters
  - Used to design transport lattices
- Instabilities in beam centroid and/or envelope oscillations can prevent reliable transport
  - Parameter locations of instability regions should be understood and avoided in machine design/operation

Although it is *necessary* to avoid envelope and centroid instabilities in designs, it is not alone *sufficient* for effective machine operation

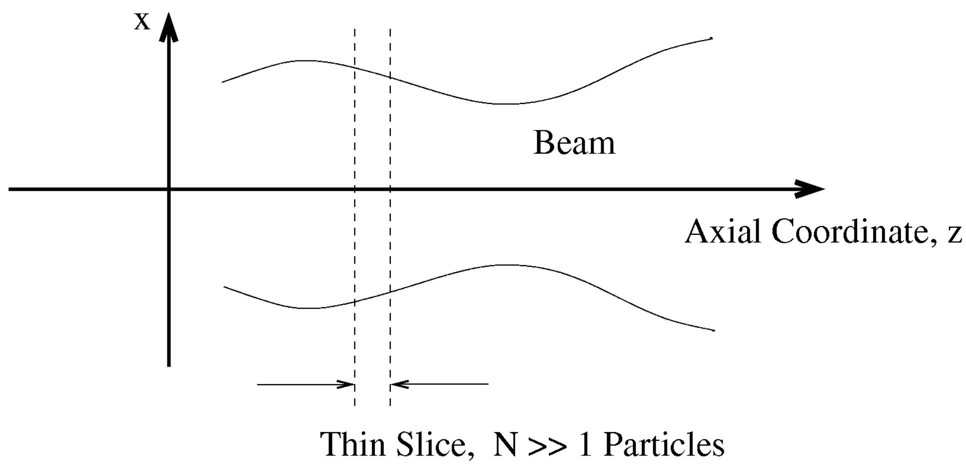
- Higher-order kinetic and fluid instabilities not expressed in the low-order envelope models can can degrade beam quality and control and must also be evaluated
  - Analysis is much harder!

Transverse Statistical Averages

Analyze centroid and envelope properties of an unbunched $(\partial/\partial z \simeq 0)$ beam

Transverse Statistical Averages:

Let N be the number of particles in a thin axial slice of the beam at axial coordinate s.



Averages can be equivalently defined in terms of the discreet particles making up the beam or the continuous model transverse Vlasov distribution function:

particles: $$\langle \cdots \rangle_\perp \equiv \frac{1}{N} \sum_{i=1}^{N} \Big|_{\text{slice}} \cdots$$

distribution: $$\langle \cdots \rangle_\perp \equiv \frac{\int d^2x_\perp \int d^2x'_\perp \ \cdots f_\perp}{\int d^2x_\perp \int d^2x'_\perp \ f_\perp}$$

$f_\perp(x, y, x', y'; p_z) =$ Particle Distribution

- Averages can be generalized to include axial momentum spread

Transverse Particle Equations of Motion

Consistent with earlier analysis take:

$$x'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} x' + \kappa_x x = -\frac{q}{m\gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial x}$$

$$y'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} y' + \kappa_y y = -\frac{q}{m\gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial y}$$

$$\nabla_\perp^2 \phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = -\frac{\rho}{\epsilon_0}$$

$$\rho = q \int d^2 x'_\perp \; f_\perp \qquad \phi|_{\text{aperture}} = 0$$

Assume:

- Unbunched beam
- No axial momentum spread
- Linear applied focusing fields described by $\kappa_x, \; \kappa_y$
- No acceleration: $\gamma_b \beta_b = \text{const}$
- Norm:

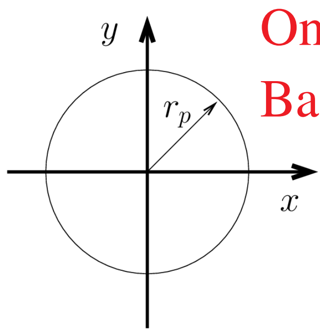
$$\lambda = q \int d^2 x'_\perp \; f_\perp$$

$$= \text{Line Charge} = \text{const}$$

Various apertures are possible influence solution for ϕ. Some simple examples:

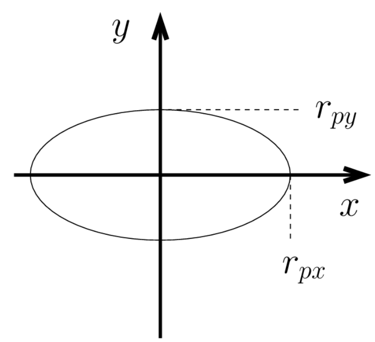
Round Pipe

Only Cover:
Bad enough



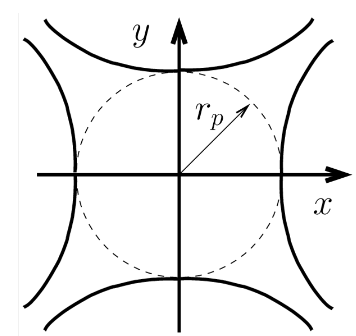
Linac magnetic quadrupoles, acceleration cells, ...

Elliptical Pipe



In rings with dispersion: in drifts, magnetic optics, ...

Hyperbolic Sections

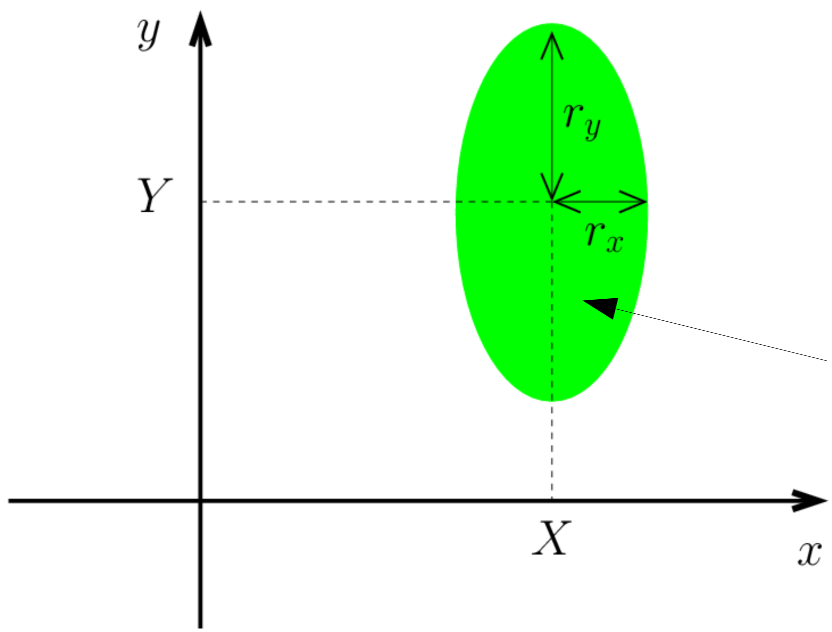


Electric quadrupoles

Distribution Assumptions

To lowest order, linearly focused intense beams are expected to be nearly uniform in density within the core of the beam out to an spatial edge where the density falls rapidly to zero. Simplest approx but also physically motivated.

- Due to Debye screening: weaker space-charge can have other evolving profiles



Charge conservation requires:

$$\lambda = \text{const}$$

Uniform density within beam:

$$\rho = \frac{\lambda}{\pi r_x r_y} \neq \text{const}$$

$$\boxed{\begin{aligned} \rho(x,y) &= q\int d^2x'_\perp\; f_\perp \;\simeq \begin{cases} \dfrac{\lambda}{\pi r_x r_y}, & (x-X)^2/r_x^2 + (y-Y)^2/r_y^2 < 1 \\ 0, & (x-X)^2/r_x^2 + (y-Y)^2/r_y^2 > 1 \end{cases} \\ \lambda &= q\int d^2x_\perp \int d^2x'_\perp\; f_\perp \;= \int d^2x\;\rho \;= \text{const} \end{aligned}}$$

Comments:

- Nearly uniform density out to a sharp spatial beam edge expected for near equilibrium structure beam with strong space-charge due to Debye screening
  - See USPAS course notes: Beam Physics with Intense Space-Charge
- Simulations support that uniform density model is a good approximation for stable non-equilibrium beams when space-charge is high
  - Variety of initial distributions launched and, where stable, rapidly relax to a fairly uniform charge density core
  - Low order core oscillations may persist with little problem evident
  - See USPAS course notes: Beam Physics with Intense Space-Charge
- Assumption of a fixed form of distribution essentially closes the infinite hierarchy of moments that are needed to describe a general beam distribution
  - Need only describe shape/edge and center for uniform density beam to fully specify the distribution!
  - Analogous to closures of fluid theories using assumed equations of state etc.

Self-Field Calculation

Temporarily, we will consider an *arbitrary* beam charge distribution within an arbitrary aperture to formulate the problem.

Electrostatic field of a line charge in free-space

$$\boxed{\mathbf{E}_\perp = \frac{\lambda_0}{2\pi\epsilon_0}\frac{(\mathbf{x}_\perp - \tilde{\mathbf{x}})}{|\mathbf{x}_\perp - \tilde{\mathbf{x}}|^2}}$$

$\lambda_0 =$ line charge

$\mathbf{x}_\perp = \tilde{\mathbf{x}} =$ coordinate of charge

Resolve the field of the beam into direct (free space) and image terms:

$$\boxed{\mathbf{E}^s_\perp = -\frac{\partial\phi}{\partial\mathbf{x}_\perp} = \mathbf{E}^d_\perp + \mathbf{E}^i_\perp}$$

and superimpose free-space solutions for direct and image contributions

Direct Field

$$\mathbf{E}^d_\perp(\mathbf{x}_\perp) = \frac{1}{2\pi\epsilon_0}\int d^2\tilde{x}_\perp \, \frac{\rho(\tilde{\mathbf{x}}_\perp)(\mathbf{x}_\perp - \tilde{\mathbf{x}}_\perp)}{|\mathbf{x}_\perp - \tilde{\mathbf{x}}_\perp|^2}$$

$\rho(\mathbf{x}_\perp) =$ beam charge density

Image Field

$$\mathbf{E}^i_\perp(\mathbf{x}_\perp) = \frac{1}{2\pi\epsilon_0}\int d^2\tilde{x}_\perp \, \frac{\rho^i(\tilde{\mathbf{x}}_\perp)(\mathbf{x}_\perp - \tilde{\mathbf{x}}_\perp)}{|\mathbf{x}_\perp - \tilde{\mathbf{x}}_\perp|^2}$$

$\rho^i(\mathbf{x}_\perp) =$ beam image charge density induced on/outside aperture

// Aside: 2D Field of Line-Charges in Free-Space

$$\nabla_\perp \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad\qquad \rho(r) = \lambda \frac{\delta(r)}{2\pi r}$$

Line charge at origin, apply Gauss' Law to obtain the field as a function of the radial coordinate *r* :

$$E_r = \frac{\lambda}{2\pi\epsilon_0 r} \qquad\qquad \mathbf{E}_\perp = \hat{\mathbf{r}} E_r$$

For a line charge at $\mathbf{x}_\perp = \tilde{\mathbf{x}}_\perp$, shift coordinates and employ vector notation:

$$\boxed{\mathbf{E}_\perp = \frac{\lambda}{2\pi\epsilon_0} \frac{\mathbf{x}_\perp - \tilde{\mathbf{x}}_\perp}{|\mathbf{x}_\perp - \tilde{\mathbf{x}}_\perp|^2}}$$

Use this and linear superposition for the field due to direct and image charges

- Metallic aperture replaced by collection of images external to the aperture in free-space to calculate consistent fields interior to the aperture

$$\boxed{\mathbf{E}_\perp = \frac{1}{2\pi\epsilon_0} \int d^2x_\perp \; \rho(\tilde{\mathbf{x}}_\perp) \frac{\mathbf{x}_\perp - \tilde{\mathbf{x}}_\perp}{|\mathbf{x}_\perp - \tilde{\mathbf{x}}_\perp|^2}}$$

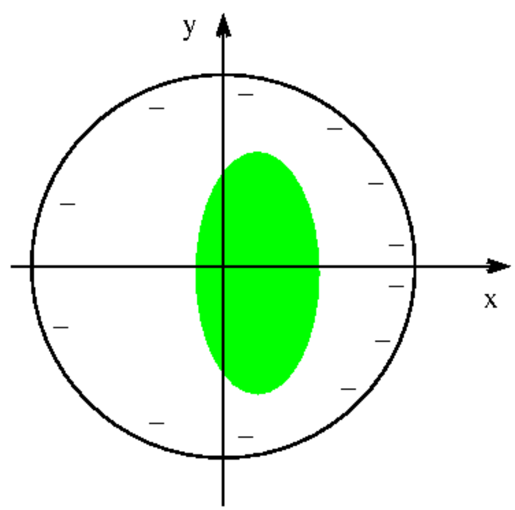
//

Comment on Image Fields

Actual charges on the conducting aperture are induced on a thin (surface charge density) layer on the inner aperture surface. In the method of images, these are replaced by a distribution of charges outside the aperture in vacuum that meet the conducting aperture boundary conditions

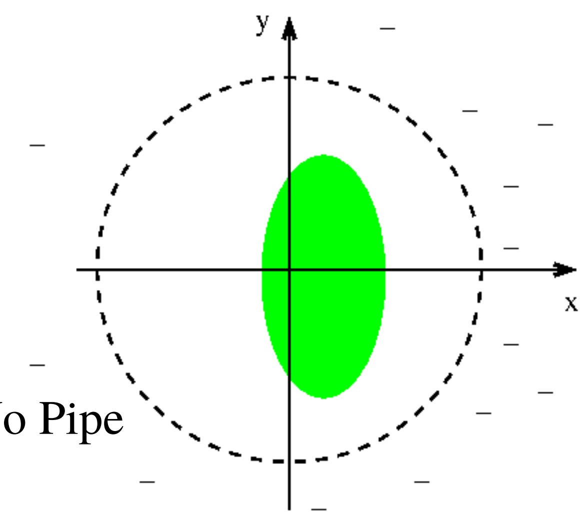
- Field within aperture can be calculated using the images in vacuum
- Induced charges on the inner aperture often called "image charges"
- Magnitude of induced charge on aperture is equal to beam charge and the total charge of the images

Physical



- No pipe
- Schematic only (really continuous image dist)

Images

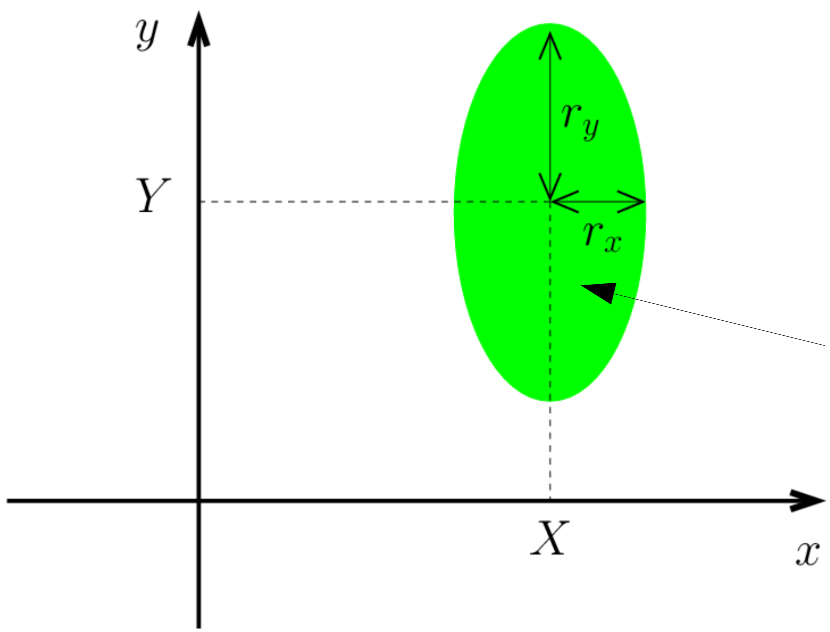


Direct Field:

The direct field solution for a uniform density beam in free-space is:

- This in NOT a trivial calculation

See USPAS course notes for Beam Physics with Intense Space-Charge



Uniform density in beam:

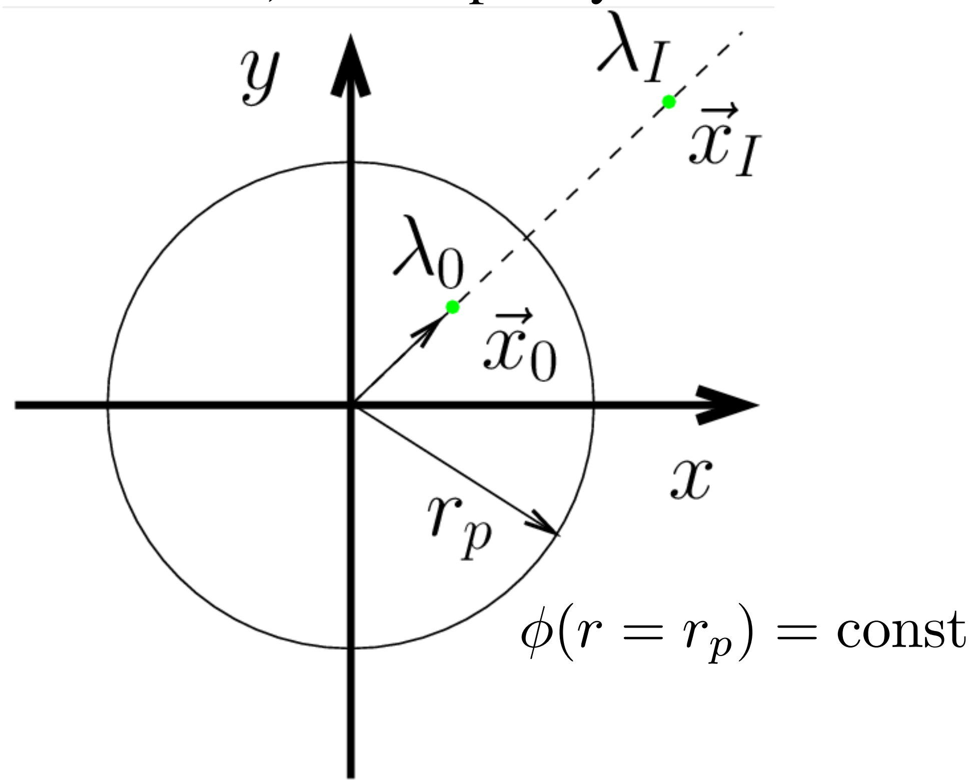
$$\rho = \frac{\lambda}{\pi r_x r_y} = \text{const}$$

$$\boxed{\begin{aligned} E_x^d &= \frac{\lambda}{\pi\epsilon_0}\frac{x-X}{(r_x+r_y)r_x} \\ E_y^d &= \frac{\lambda}{\pi\epsilon_0}\frac{y-Y}{(r_x+r_y)r_y} \end{aligned}}$$

Expressions are valid only within the elliptical density beam -- where they will be applied in taking averages

Image Field:

Image structure depends on the aperture. Assume a round pipe (most common case) for simplicity.



$$\lambda_I = -\lambda_0 \qquad \text{image charge}$$

$$\mathbf{x}_I = \frac{r_p^2}{|\mathbf{x}_0|^2}\mathbf{x}_0 \qquad \text{image location}$$

~~Will be derived in the the problem sets.~~

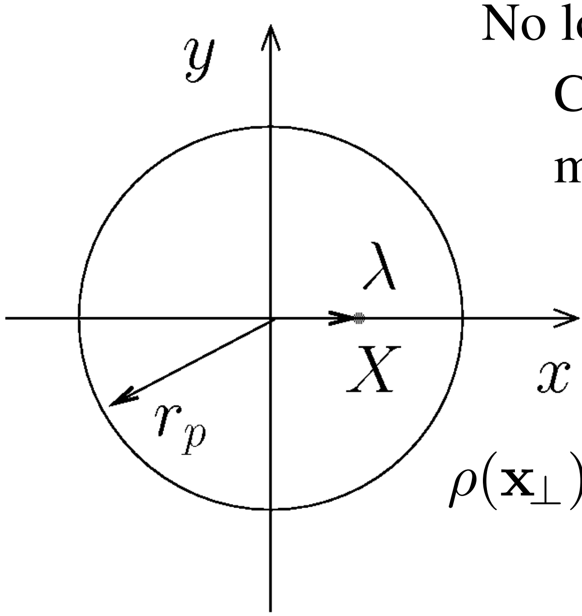
Superimpose all images of beam to obtain the image contribution in aperture:

$$\boxed{\mathbf{E}^i_\perp(\mathbf{x}_\perp) = -\frac{1}{2\pi\epsilon_0}\int_{\text{pipe}} d^2\tilde{x}_\perp \, \frac{\rho(\tilde{\mathbf{x}}_\perp)(\mathbf{x}_\perp - r_p^2\tilde{\mathbf{x}}_\perp/|\tilde{\mathbf{x}}_\perp|^2)}{|\mathbf{x}_\perp - r_p^2\tilde{\mathbf{x}}_\perp/|\tilde{\mathbf{x}}_\perp|^2|^2}}$$

- Difficult to calculate even for ρ corresponding to a uniform density beam

Examine limits of the image field to build intuition on the range of properties:

1) Line charge along *x*-axis:



No loss in generality:

Can always choose coordinates to make charge lie on axis

$$\rho(\mathbf{x}_\perp) = \lambda\delta(\mathbf{x}_\perp - X\hat{\mathbf{x}})$$

$$\mathbf{E}^i_\perp = \frac{\lambda^i}{2\pi\epsilon_0}\frac{\mathbf{x}_\perp - \mathbf{x}^i_\perp}{|\mathbf{x}_\perp - \mathbf{x}^i_\perp|^2}$$

$$\lambda^i = -\lambda$$

$$\mathbf{x}^i_\perp = \frac{r_p^2}{X}\hat{\mathbf{x}}$$

Plug this density in the image charge expression for a round-pipe aperture:

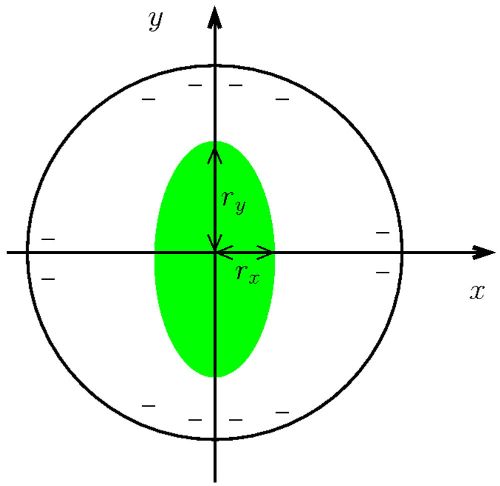
- Need only evaluate at $\mathbf{x}_\perp = X\hat{\mathbf{x}}$ since beam is at that location

$$\boxed{\mathbf{E}^i_\perp(\mathbf{x}_\perp = X\hat{\mathbf{x}}) = \frac{\lambda}{2\pi\epsilon_0(r_p^2/X - X)}\hat{\mathbf{x}}}$$

- Generates nonlinear field at position of direct charge
- Field creates attractive force between direct and image charge
  - Therefore image charge should be expected to "drag" centroid further off
  - Amplitude of centroid oscillations expected to increase if not corrected (steering)

2) Centered, uniform density elliptical beam:

SKIP: For your information only. More complicated



$$\rho(\mathbf{x}_\perp) = \begin{cases} \frac{\lambda}{\pi r_x r_y}, & x^2/r_x^2 + y^2/r_y^2 < 1 \\ 0, & x^2/r_x^2 + y^2/r_y^2 > 1 \end{cases}$$

Expand using complex coordinates starting from the general image expression:

- Image field is in vacuum aperture so complex methods help calculation

$$\underline{E}^{i*} = E_x^i - iE_y^i = \sum_{n=2,4,\cdots}^{\infty} \underline{c}_n \underline{z}^{n-1} \qquad \underline{c}_n = \frac{1}{2\pi\epsilon_0} \int_{\text{pipe}} d^2x_\perp \; \rho(\mathbf{x}_\perp) \frac{(x-iy)^n}{r_p^{2n}}$$

$$\underline{z} = x + iy \qquad i = \sqrt{-1} \qquad = \frac{\lambda n!}{2\pi\epsilon_0 2^n (n/2+1)!(n/2)!} \left(\frac{r_x^2 - r_y^2}{r_p^4} \right)^{n/2}$$

The linear (n = 2) components of this expansion give:

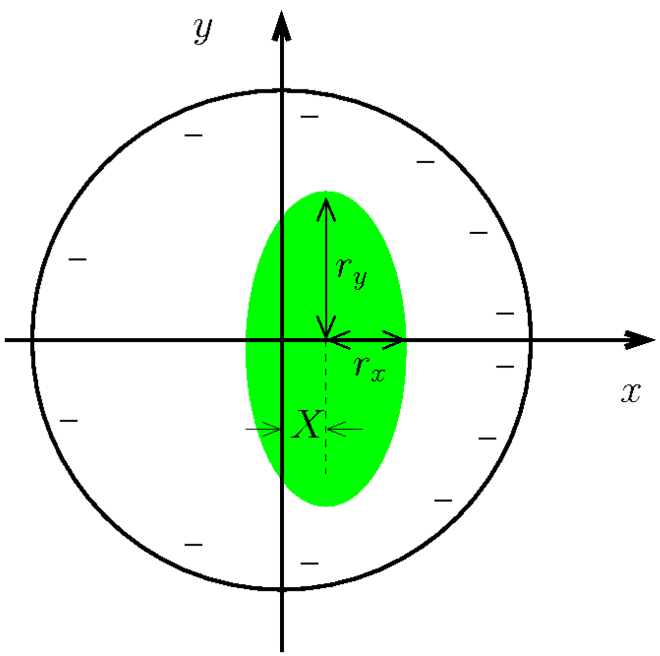
$$\boxed{E_x^i = \frac{\lambda}{8\pi\epsilon_0} \frac{r_x^2 - r_y^2}{r_p^4} x, \qquad E_y^i = -\frac{\lambda}{8\pi\epsilon_0} \frac{r_x^2 - r_y^2}{r_p^4} y}$$

- Rapidly vanish (higher order n terms more rapidly) as beam becomes more round

3) Uniform density elliptical beam with a small displacement along the *x*-axis:

SKIP: For your information only: Very Complicated – Much worse than previous limits

$Y = 0 \qquad |X|/r_p \ll 1$



Expand using complex coordinates starting from the general image expression:

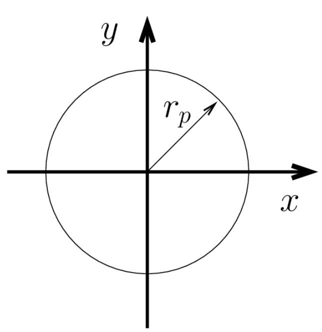
- Complex coordinates help simplify very messy calculation

  E.P. Lee, E. Close, and L. Smith, Nuclear Instruments and Methods, 1126 (1987)

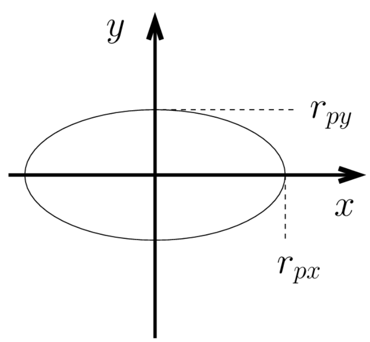
Comments on images:

- Sign is generally such that it will tend to increase beam centroid displacements
  - Also (usually) weak linear focusing corrections for an elliptical beam
- Can be very difficult to calculate explicitly
  - Even for simple case of circular pipe
  - Special cases of simple geometry and case formulas help clarify scaling
  - Generally suppress by making the beam small relative to characteristic aperture dimensions and keeping the beam steered near-axis
  - Simulations typically applied
- Depend strongly on the aperture geometry
  - Generally varies as a function of *s* in the machine aperture due to changes in accelerator lattice elements and/or as beam symmetries evolve

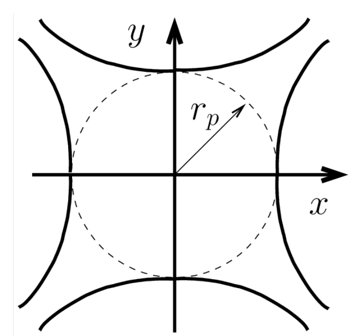
Round Pipe



Elliptical Pipe



Hyperbolic Sections



Centroid and Envelope equations of motion for a uniform density elliptical beam

Consistent with the assumed structure of the distribution (uniform density elliptical beam), denote:

Beam Centroid:

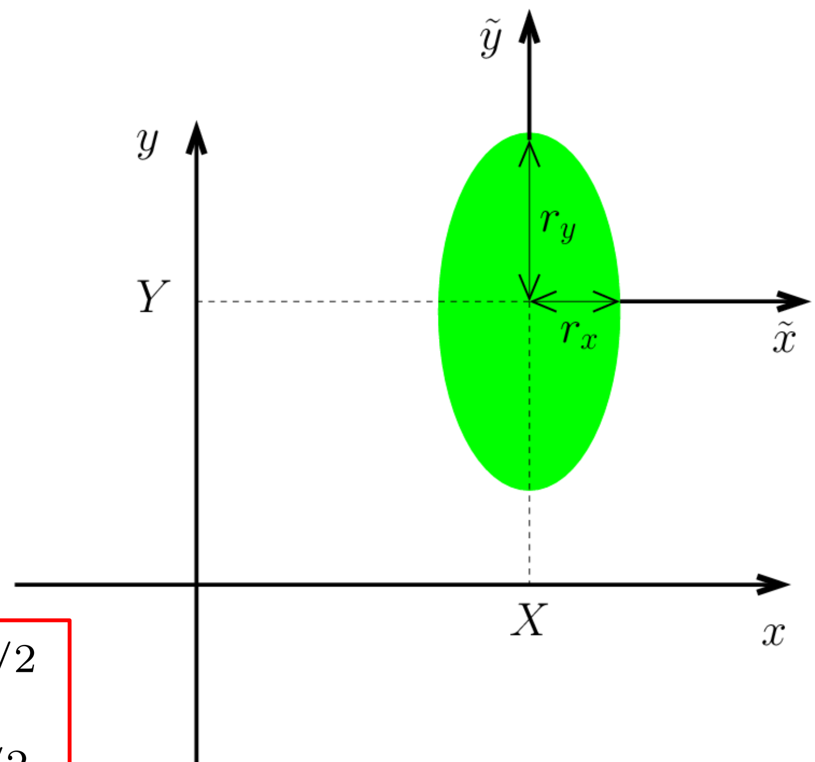
$$X \equiv \langle x \rangle_\perp \qquad X' = \langle x' \rangle_\perp$$
$$Y \equiv \langle y \rangle_\perp \qquad Y' = \langle y' \rangle_\perp$$

Coordinates with respect to centroid:

$$\tilde{x} \equiv x - X \qquad \tilde{x}' = x' - X'$$
$$\tilde{y} \equiv y - Y \qquad \tilde{y}' = y' - Y'$$

Envelope Edge Radii:

$$r_x \equiv 2\sqrt{\langle \tilde{x}^2 \rangle_\perp} \qquad r'_x = 2\langle \tilde{x}\tilde{x}' \rangle_\perp / \langle \tilde{x}^2 \rangle_\perp^{1/2}$$
$$r_y \equiv 2\sqrt{\langle \tilde{y}^2 \rangle_\perp} \qquad r'_y = 2\langle \tilde{y}\tilde{y}' \rangle_\perp / \langle \tilde{y}^2 \rangle_\perp^{1/2}$$



With the *assumed* uniform elliptical beam, all moments can be calculated in terms of: $X, \; Y \qquad r_x, \; r_y$

- Such truncations follow whenever the form of the distribution is "frozen"

Centroid equations of motion

Derive centroid equations: First use the self-field resolution for a uniform density beam, then the equations of motion for a particle within the beam are:

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x x = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$

$$-\frac{\partial\phi}{\partial x} = E_x^d + E_x^i = \frac{\lambda}{\pi\epsilon_0}\frac{x-X}{(r_x+r_y)r_x} + E_x^i$$

$$\boxed{\begin{aligned} x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x x - \frac{2Q}{(r_x+r_y)r_x}(x-X) &= \frac{q}{m\gamma_b^3\beta_b^2c^2}E_x^i \\ y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' + \kappa_y y - \frac{2Q}{(r_x+r_y)r_y}(y-Y) &= \frac{q}{m\gamma_b^3\beta_b^2c^2}E_y^i \end{aligned}}$$

Direct Terms

Image Terms

Dimensionless Perveance Q measures space-charge strength:

$$Q \equiv \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2c^2}$$

- Constant if beam coasting with no acceleration
- Small number typical: 10^-2 near injector to 10^-6 or smaller at higher energies

Take average of x-equation over the distribution

$$Q \equiv \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2 c^2}$$

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x x - \frac{2Q}{(r_x+r_y)r_x}(x - X) = \frac{q}{m\gamma_b^3\beta_b^2 c^2}E_x^i$$

$$\langle x''\rangle_\perp + \langle\frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x'\rangle_\perp + \langle\kappa_x x\rangle_\perp - \langle\frac{2Q}{(r_x+r_y)r_x}(x - X)\rangle_\perp = \langle\frac{q}{m\gamma_b^3\beta_b^2 c^2}E_x^i\rangle_\perp$$

$$\langle x\rangle_\perp'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\langle x'\rangle_\perp + \kappa_x\langle x\rangle_\perp - \frac{2Q}{(r_x+r_y)r_x}\langle x - X\rangle_\perp$$

$$= \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2 c^2}\left[\frac{2\pi\epsilon_0}{\lambda}\right]\langle E_x^i\rangle_\perp$$

Use:

$$X = \langle x\rangle_\perp$$

$$\langle x - X\rangle_\perp = X - X = 0$$

Centroid Equations: (y-equation similar)

$$\Longrightarrow \boxed{\begin{aligned} X'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}X' + \kappa_x X &= Q\left[\frac{2\pi\epsilon_0}{\lambda}\langle E_x^i\rangle_\perp\right] \\ Y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}Y' + \kappa_y Y &= Q\left[\frac{2\pi\epsilon_0}{\lambda}\langle E_y^i\rangle_\perp\right] \end{aligned}}$$

Note: the electric image field will cancel the coefficient $2\pi\epsilon_0/\lambda$

$$\mathbf{E}_\perp^i = \frac{1}{2\pi\epsilon_0}\int d^2\tilde{x}_\perp\,\frac{\rho^i(\tilde{\mathbf{x}}_\perp)(\mathbf{x}_\perp - \tilde{\mathbf{x}}_\perp)}{|\mathbf{x}_\perp - \tilde{\mathbf{x}}_\perp|^2}$$

- $\langle E_x^i\rangle_\perp$ will generally depend on: X, Y and r_x, r_y

Example Evolution Centroid Equations of Motion
Single Particle Limit: Oscillation and Stability Properties

Neglect image charge terms, then the centroid equation of motion becomes:

$$\boxed{\begin{aligned} X'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}X' + \kappa_x X = 0 \\ Y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}Y' + \kappa_y Y = 0 \end{aligned}}$$

- Usual Hill's equation with acceleration term
- Single particle form and can apply usual results: phase amplitude methods, Courant-Snyder invariants, and stability bounds, ...

Assume that applied lattice focusing is tuned for constant phase advances with normalized coordinates (effective κ_x, κ_y) and/or that acceleration is weak and can be neglected. Then single particle stability results give immediately:

$$\begin{aligned} \frac{1}{2}|\text{Tr }\mathbf{M}_x(s_i + L_p|s_i)| \leq 1 \\ \frac{1}{2}|\text{Tr }\mathbf{M}_y(s_i + L_p|s_i)| \leq 1 \end{aligned} \quad \Longleftrightarrow \quad \boxed{\begin{aligned} \sigma_{0x} < 180^\circ \\ \sigma_{0y} < 180^\circ \end{aligned}}$$

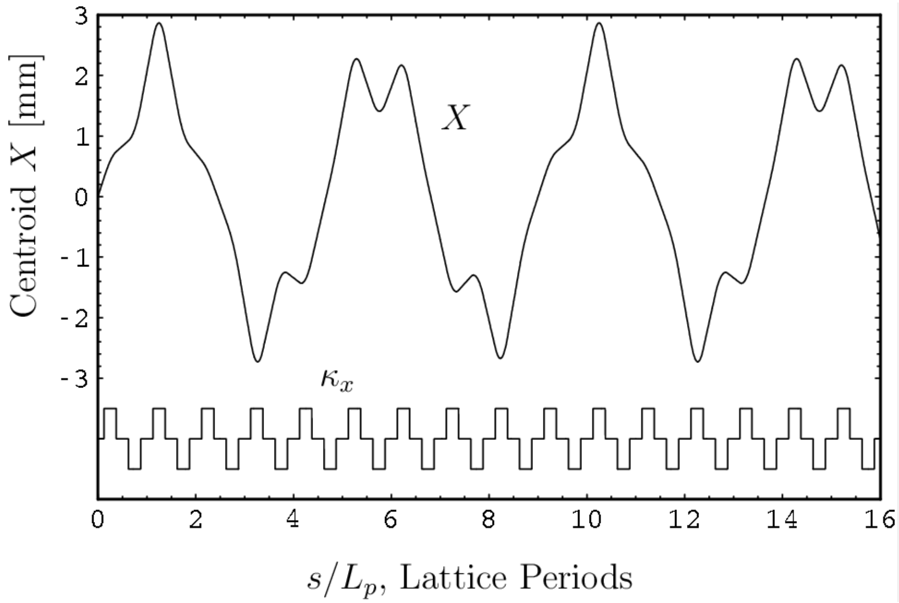
centroid stability

/// Example: FODO channel centroid evolution for a coasting beam

Mid-drift launch:

$X(0) = 0$ mm

$X'(0) = 1$ mrad



FODO quad lattice/beam parameters:

$\gamma_b \beta_b = \text{const}$

$\sigma_{0x} = 80^\circ$

$L_p = 0.5$ m

$\eta = 0.5$

- Centroid exhibits expected characteristic stable betatron oscillations
  - Stable so oscillation amplitude does not grow
  - Courant-Snyder invariant (i.e, initial centroid phase-space area set by initial conditions) and betatron function can be used to bound oscillation
- Motion in *y*-plane analogous

///

Designing a lattice for single particle stability by limiting undepressed phases advances to less that 180 degrees per period means that the centroid will be stable

- Situation could be modified in very extreme cases due to image couplings

Effects of Image Charges: let oscillation be along x-axis

Model the beam as a displaced line-charge in a circular aperture. Then using the previously derived image charge field, the equations of motion reduce to:

$$\boxed{X'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}X' + \kappa_x X = \frac{QX}{r_p^2 - X^2}}$$

$$X'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}X' + \kappa_x X = Q\left[\frac{2\pi\epsilon_0}{\lambda}\langle E_x^i\rangle_\perp\right]$$

$$\mathbf{E}_\perp^i(\mathbf{x}_\perp = X\hat{\mathbf{x}}) = \frac{\lambda}{2\pi\epsilon_0(r_p^2/X - X)}\hat{\mathbf{x}}$$

$$\frac{QX}{r_p^2 - X^2} \simeq \frac{Q}{r_p^2}X + \frac{Q}{r_p^4}X^3$$

linear correction

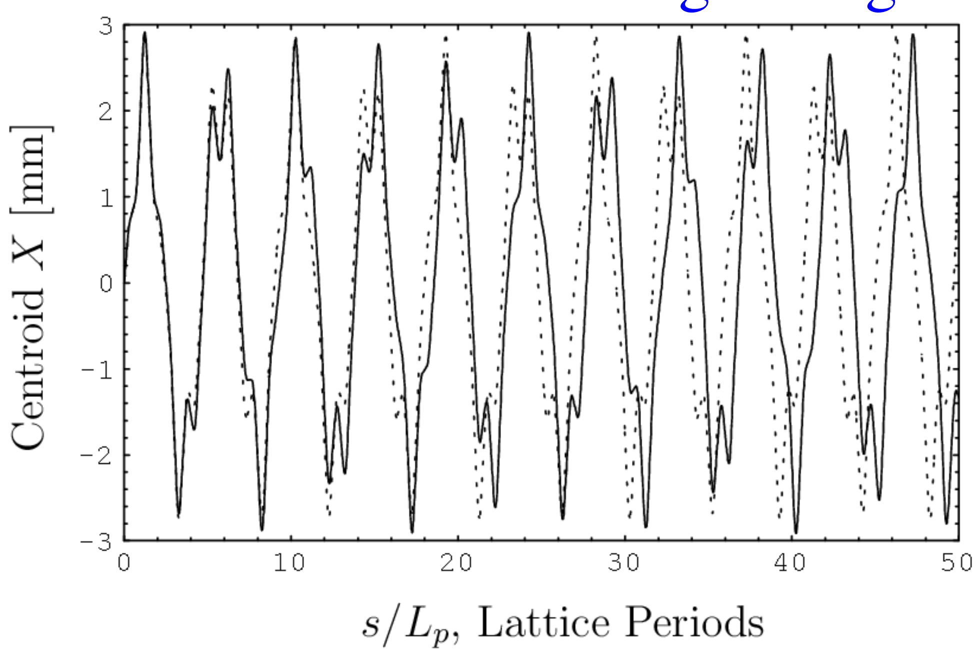
Nonlinear correction (smaller)

Example: FODO channel centroid with image charge corrections

$r_p = 30$ mm

$Q = 2 \times 10^{-4}$

same lattice as previous



solid – with images

dashed – no images

Main effect of images is typically an accumulated phase error of the centroid orbit

- This will complicate extrapolations of errors over many lattice periods

Control by:

- Keeping centroid displacements X, Y small by correcting
- Make aperture (pipe radius r_p) larger

Comments:

- Images contributions to centroid excursions typically less problematic than misalignment errors in focusing elements
- More detailed analysis show that the coupling of the envelope radii $r_x, \ r_y$ to the centroid evolution in X, Y is often weak
- Over long path lengths many nonlinear terms can also influence oscillation phase

Envelope equations of motion

To derive equations of motion for the envelope radii, first subtract the centroid equations from the particle equations of motion ($\tilde{x} \equiv x - X$) to obtain:

$$\boxed{\begin{aligned} \tilde{x}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{x}' + \kappa_x\tilde{x} - \frac{2Q\tilde{x}}{(r_x+r_y)r_x} &= \frac{q}{m\gamma_b^3\beta_b^2c^2}\left[E_x^i - \langle E_x^i\rangle_\perp\right] \\ \tilde{y}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{y}' + \kappa_y\tilde{y} - \frac{2Q\tilde{y}}{(r_x+r_y)r_x} &= \frac{q}{m\gamma_b^3\beta_b^2c^2}\left[E_y^i - \langle E_y^i\rangle_\perp\right] \end{aligned}}$$

Differentiate the equation for the envelope radius twice (y-equations analogous):

$$r_x \equiv 2\langle\tilde{x}^2\rangle_\perp^{1/2} \quad\Longrightarrow\quad r_x' = \frac{2\langle\tilde{x}\tilde{x}'\rangle_\perp}{\langle\tilde{x}^2\rangle_\perp^{1/2}} = \frac{4\langle\tilde{x}\tilde{x}'\rangle_\perp}{r_x}$$

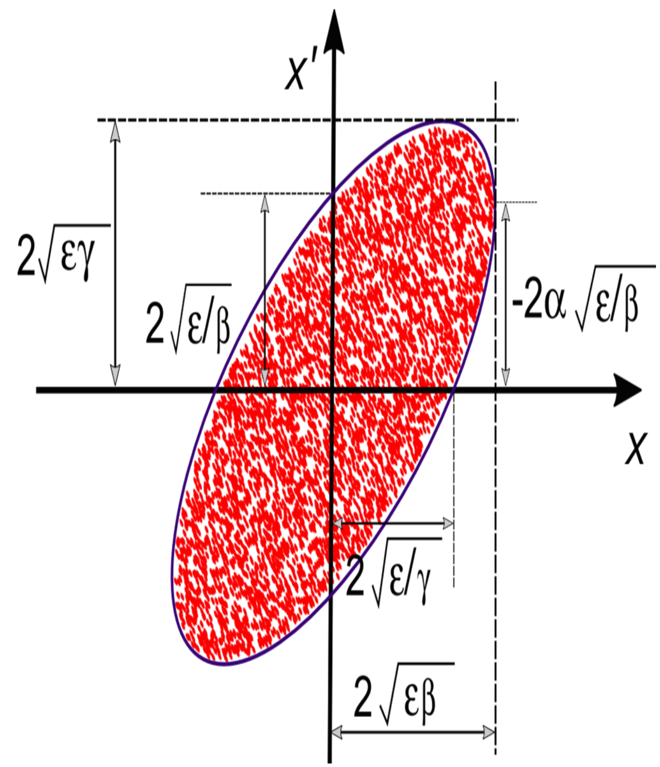
$$\begin{aligned} r_x'' &= \frac{2\langle\tilde{x}\tilde{x}''\rangle_\perp}{\langle\tilde{x}^2\rangle_\perp^{1/2}} + \frac{2\langle\tilde{x}'^2\rangle_\perp}{\langle\tilde{x}^2\rangle_\perp^{1/2}} - \frac{2\langle\tilde{x}\tilde{x}'\rangle_\perp^2}{\langle\tilde{x}^2\rangle_\perp^{3/2}} \\ &= 4\frac{\langle\tilde{x}\tilde{x}''\rangle_\perp}{[2\langle\tilde{x}^2\rangle_\perp^{1/2}]} + \frac{16\left[\langle\tilde{x}^2\rangle_\perp\langle\tilde{x}'^2\rangle_\perp - \langle\tilde{x}\tilde{x}'\rangle_\perp^2\right]}{[2\langle\tilde{x}^2\rangle_\perp^{1/2}]^3} \\ &= 4\frac{\langle\tilde{x}\tilde{x}''\rangle_\perp}{r_x} + \frac{16\left[\langle\tilde{x}^2\rangle_\perp\langle\tilde{x}'^2\rangle_\perp - \langle\tilde{x}\tilde{x}'\rangle_\perp^2\right]}{r_x^3} \end{aligned}$$

Define a statistical measure of beam phase-space area with the rms edge emittance:

- Form motivated in graduate Accelerator Physics for statistical measure of phase-space area of the beam
- Commonly used in lab diagnostics and simulations

Rms Edge Emittance

$$\boxed{\varepsilon_x \equiv 4\varepsilon_{x,\mathrm{rms}} \equiv 4\left[\langle\tilde{x}^2\rangle_\perp \langle\tilde{x}'^2\rangle_\perp - \langle\tilde{x}\tilde{x}'\rangle_\perp^2\right]^{1/2}}$$



Ref: Comsol Blog

- Units commonly given in mm-mrad
- $\pi\varepsilon_x$ is *x*-*x*' area for uniformly filled ellipse
- Expect (homework) that $\varepsilon_x = \mathrm{const}$ for linear forces

Then we have:

$$\boxed{\begin{aligned} r_x'' &= 4\frac{\langle\tilde{x}\tilde{x}''\rangle_\perp}{r_x} + \frac{16[\langle\tilde{x}^2\rangle_\perp\langle\tilde{x}'^2\rangle_\perp - \langle\tilde{x}\tilde{x}'\rangle_\perp^2]}{r_x^3} \\ &= 4\frac{\langle\tilde{x}\tilde{x}''\rangle_\perp}{r_x} + \frac{\varepsilon_x^2}{r_x^3} \end{aligned}}$$

and employ the equations of motion to eliminate $\tilde{x}''$ in $\langle\tilde{x}\tilde{x}''\rangle_\perp$ with the following steps

Using the equation of motion:

$$\tilde{x}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{x}' + \kappa_x\tilde{x} - \frac{2Q\tilde{x}}{(r_x+r_y)r_x} = \frac{q}{m\gamma_b^3\beta_b^2c^2}\left[E_x^i - \langle E_x^i\rangle_\perp\right]$$

$$\langle\tilde{x}\tilde{x}''\rangle_\perp + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\langle\tilde{x}\tilde{x}'\rangle_\perp + \kappa_x\langle\tilde{x}^2\rangle_\perp - \frac{2Q\langle\tilde{x}^2\rangle_\perp}{(r_x+r_y)r_x} = \frac{q}{m\gamma_b^3\beta_b^2c^2}\left[\langle\tilde{x}E_x^i\rangle_\perp - \langle\tilde{x}\langle E_x^i\rangle_\perp\rangle_\perp\right]$$

But:

$$\langle\tilde{x}\langle E_x^i\rangle_\perp\rangle_\perp = \overbrace{\langle\tilde{x}\rangle_\perp}^{0}\langle E_x^i\rangle_\perp = 0$$

Giving when using the edge definition: $r_x \equiv 2\langle\tilde{x}^2\rangle_\perp^{1/2} \quad \longrightarrow \quad \begin{array}{l} \langle\tilde{x}^2\rangle_\perp = \dfrac{r_x^2}{4} \\ \langle\tilde{x}\tilde{x}'\rangle_\perp = \dfrac{r_x r_x'}{4} \end{array}$

$$\langle\tilde{x}\tilde{x}''\rangle_\perp + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\langle\tilde{x}\tilde{x}'\rangle_\perp + \kappa_x\langle\tilde{x}^2\rangle_\perp - \frac{2Q\langle\tilde{x}^2\rangle_\perp}{(r_x+r_y)r_x} = \frac{q}{m\gamma_b^3\beta_b^2c^2}\langle\tilde{x}E_x^i\rangle_\perp$$

$$\langle\tilde{x}\tilde{x}''\rangle_\perp + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\frac{r_x r_x'}{4} + \kappa_x\frac{r_x^2}{4} - \frac{Qr_x/2}{r_x+r_y} = \frac{q}{m\gamma_b^3\beta_b^2c^2}\langle\tilde{x}E_x^i\rangle_\perp$$

Using this $\langle\tilde{x}\tilde{x}''\rangle_\perp$ moment expression in the equation

$$\mathrm{r}_x'' = 4\frac{\langle\tilde{x}\tilde{x}''\rangle_\perp}{r_x} + \frac{\varepsilon_x^2}{r_x^3}$$

then gives the envelope equation with the image charge couplings as:

Giving:

$$r_x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} r_x' + \kappa_x r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} = 8Q\left[\frac{\pi\epsilon_0}{\lambda}\langle\tilde{x}E_x^i\rangle_\perp\right]$$

$$r_y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} r_x' + \kappa_y r_y - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_y^2}{r_y^3} = 8Q\left[\frac{\pi\epsilon_0}{\lambda}\langle\tilde{y}E_y^i\rangle_\perp\right]$$

- Image coupling term $\langle\tilde{x}E_x^i\rangle_\perp$ will generally depend on: $X,\ Y$ and $r_x,\ r_y$

Comments on Centroid/Envelope equations:

- Image terms contain nonlinear terms that can be difficult to evaluate explicitly
  - Aperture geometry changes image correction
- The formulation is not self-consistent because a frozen form (uniform density) charge profile is assumed
- Image terms typically found (numerical modeling) to have only a weak impact on the envelope and can typically be dropped

Envelope Equations:

$$\boxed{\begin{aligned} r_x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} r_x' + \kappa_x r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} &= 0 \\ r_y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} r_x' + \kappa_y r_y - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_y^2}{r_y^3} &= 0 \end{aligned}}$$

Comments on Centroid/Envelope equations (Continued):

- Constant (normalized when accelerating) emittances are generally assumed
  Will prove in homework for coasting beam with $\gamma_b \beta_b = \text{const}$
  and linear equations of motion:

$$\varepsilon_x \equiv 4\varepsilon_{x,\text{rms}} \equiv 4\left[\langle \tilde{x}^2 \rangle_\perp \langle \tilde{x}'^2 \rangle_\perp - \langle \tilde{x}\tilde{x}' \rangle_\perp^2\right]^{1/2}$$
$$= \text{const}$$

- Homework motivates relation of emittance to phase-space area
- For strong space charge emittance terms small and limited emittance evolution does not strongly influence evolution outside of final focus

$\beta_b, \;\; \gamma_b, \;\; \lambda$ *s*-variation set by acceleration schedule

$$\varepsilon_{nx} = \gamma_b \beta_b \varepsilon_x = \text{const}$$
$$\varepsilon_{ny} = \gamma_b \beta_b \varepsilon_y = \text{const}$$

$\longrightarrow$ used to calculate $\varepsilon_x, \;\; \varepsilon_y$

$$Q = \frac{q\lambda}{2\pi m \epsilon_0 \gamma_b^3 \beta_b^2 c^2}$$

Can also vary with acceleration

Envelope Equations: Properties of Terms

The envelope equation reflects low-order force balances:

$$r_x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} r_x' + \kappa_x r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} = 0$$

$$r_y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} r_y' + \kappa_y r_y - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_y^2}{r_y^3} = 0$$

	Streaming	Applied Acceleration	Applied Focusing	Space-Charge Defocusing	Thermal Defocusing
Terms:	Inertial	Lattice	Lattice	Perveance	Emittance

The "acceleration schedule" specifies both $\gamma_b\beta_b$ and λ

then the equations are integrated with:

$$\gamma_b\beta_b\varepsilon_x = \text{const}$$
$$\gamma_b\beta_b\varepsilon_y = \text{const}$$

normalized emittance conservation

(set by initial value)

$$Q = \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2 c^2}$$

specified dimensionless perveance

Properties of Envelope Equation Terms:

Inertial: $r''_x, \quad r''_y$

Applied Focusing: $\kappa_x r_x, \quad \kappa_y r_y$ and Acceleration: $\frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} r'_x, \quad \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} r'_y$

- Analogous to single-particle orbit terms in Transverse Particle Dynamics
- Contributions to beam envelope essentially the same as in single particle case
- Have strong *s* dependence, *can be both focusing and defocusing*
  - Act only in focusing elements and acceleration gaps
  - Net tendency to damp oscillations with energy gain

Perveance: $\frac{2Q}{r_x + r_y}$ Scale $\sim \frac{1}{\text{Env. Radius}}$

- Acts continuously in *s*, *always defocusing*
- Becomes stronger (relatively to other terms) when the beam expands in cross-sectional area

Emittance: $\frac{\varepsilon_x^2}{r_x^3}$ Scale $\sim \frac{1}{(\text{Env. Radius})^3}$

- Acts continuously in *s*, *always defocusing*
- Becomes stronger (relatively to other terms) when the beam becomes small in cross-sectional area
- Scaling makes clear why it is necessary to inhibit emittance growth for applications where small spots are desired on target

Matched Envelope Solution:

Neglect acceleration $(\gamma_b \beta_b = \text{const})$:

$$\boxed{\begin{aligned} & r_x''(s) + \kappa_x(s) r_x(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_x^2}{r_x^3(s)} = 0 \\ & r_y''(s) + \kappa_y(s) r_y(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_y^2}{r_y^3(s)} = 0 \\ & r_x(s + L_p) = r_x(s) \qquad\qquad r_x(s) > 0 \\ & r_y(s + L_p) = r_y(s) \qquad\qquad r_y(s) > 0 \end{aligned}}$$

Matching involves finding specific initial conditions for the envelope to have the periodicity of the lattice:

Find Values of: $\quad$ Such That: (periodic) $\quad L_p = \text{Lattice Period}$

$$\boxed{\begin{array}{cc} r_x(s_i) & r_x'(s_i) \\ r_y(s_i) & r_y'(s_i) \end{array}} \longrightarrow \boxed{\begin{array}{cc} r_x(s_i + L_p) = r_x(s_i) & r_x'(s_i + L_p) = r_x'(s_i) \\ r_y(s_i + L_p) = r_y(s_i) & r_y'(s_i + L_p) = r_y'(s_i) \end{array}}$$

- Typically constructed with numerical root finding from estimated/guessed values
  - Can be surprisingly difficult for complicated lattices (high σ_0) with strong space-charge

Matched solution to the envelope equations reflects the symmetry of the focusing lattice and must in general be calculated numerically

Matching Condition

$$\boxed{\begin{aligned} r_x(s+L_p) &= r_x(s) \\ r_y(s+L_p) &= r_y(s) \end{aligned}}$$

Example Parameters

$$L_p = 0.5 \text{ m}, \quad \sigma_0 = 80^\circ, \quad \eta = 0.5$$

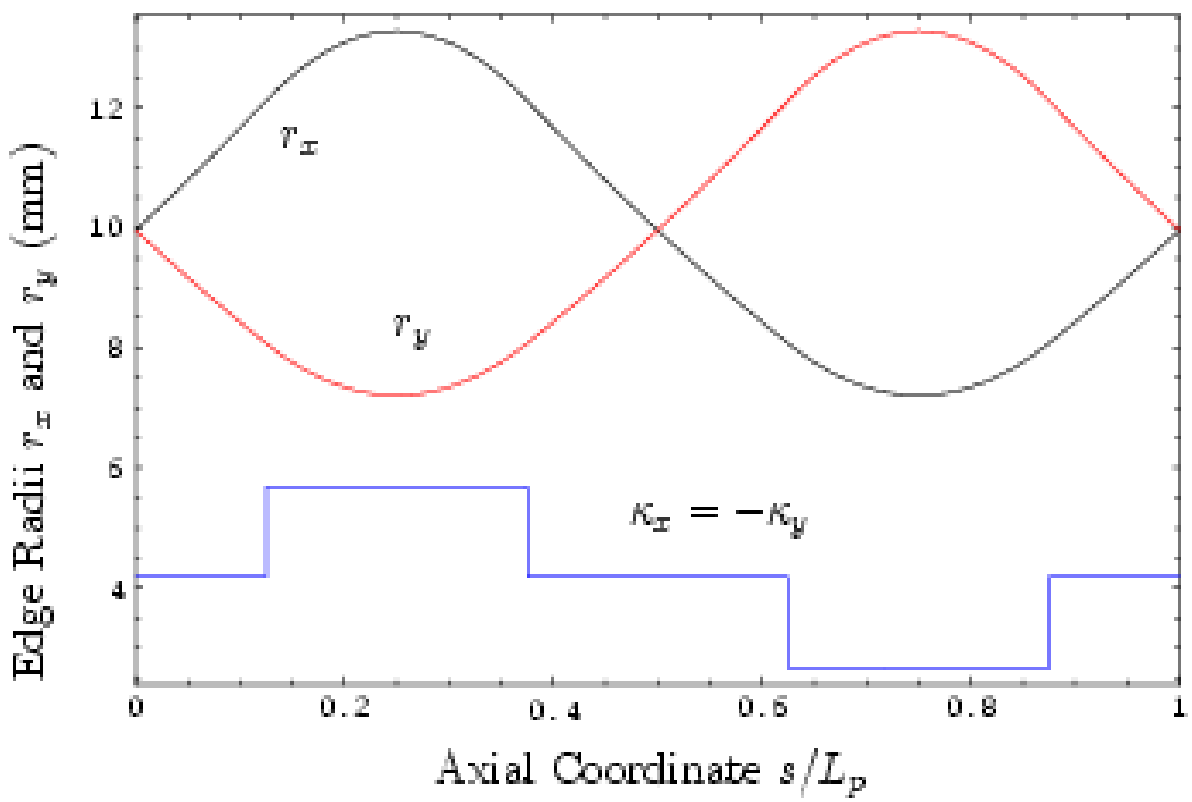
$$\varepsilon_x = \varepsilon_y = 50 \text{ mm-mrad}$$

$$Q = 6.5614 \times 10^{-4}$$

$$\Longleftrightarrow \sigma/\sigma_0 = 0.2$$

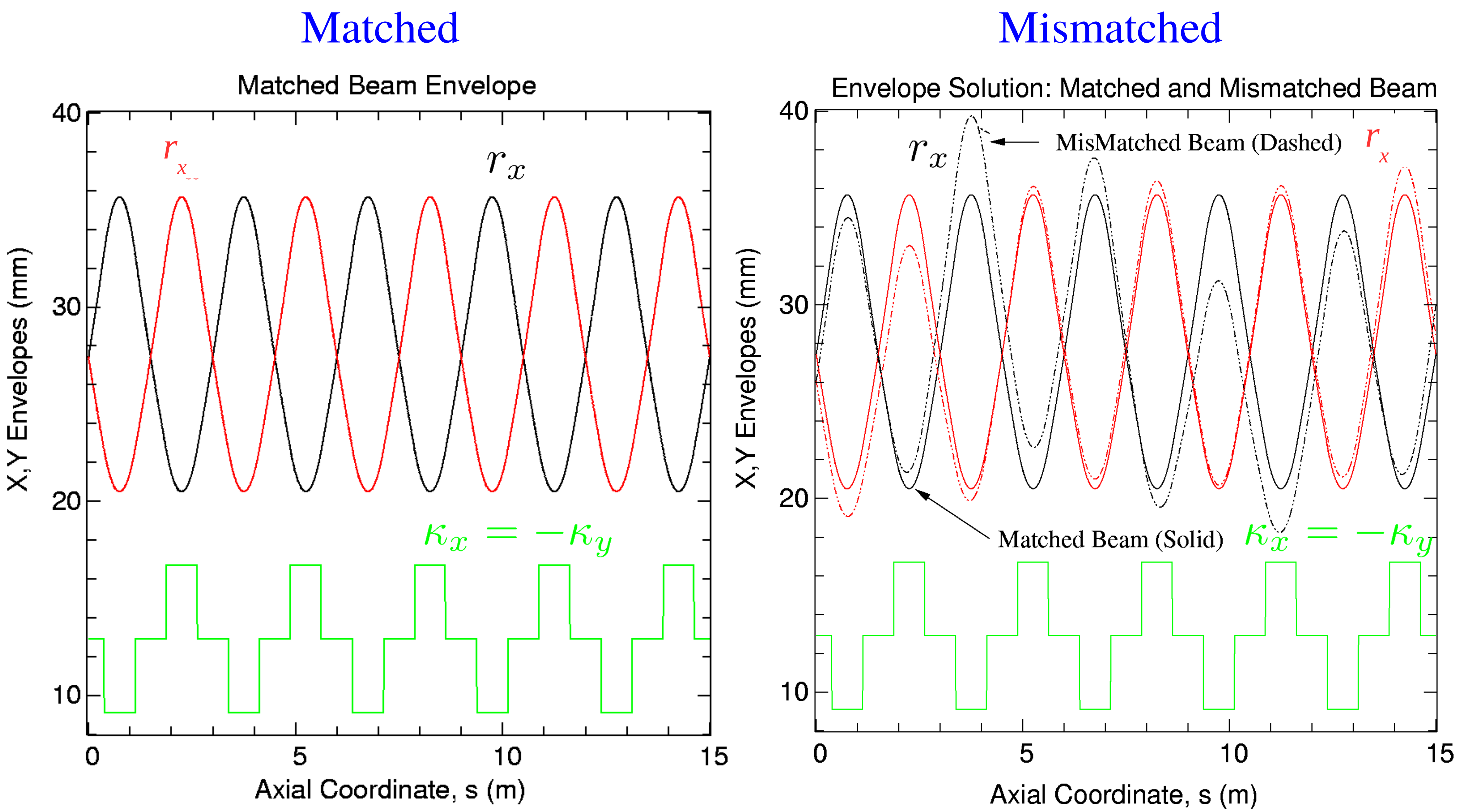
(Q large enough for SC to cancel 80% applied focus)

FODO Quadrupole Focusing



The matched beam is the most radially compact solution to the envelope equations rendering it highly important for beam transport

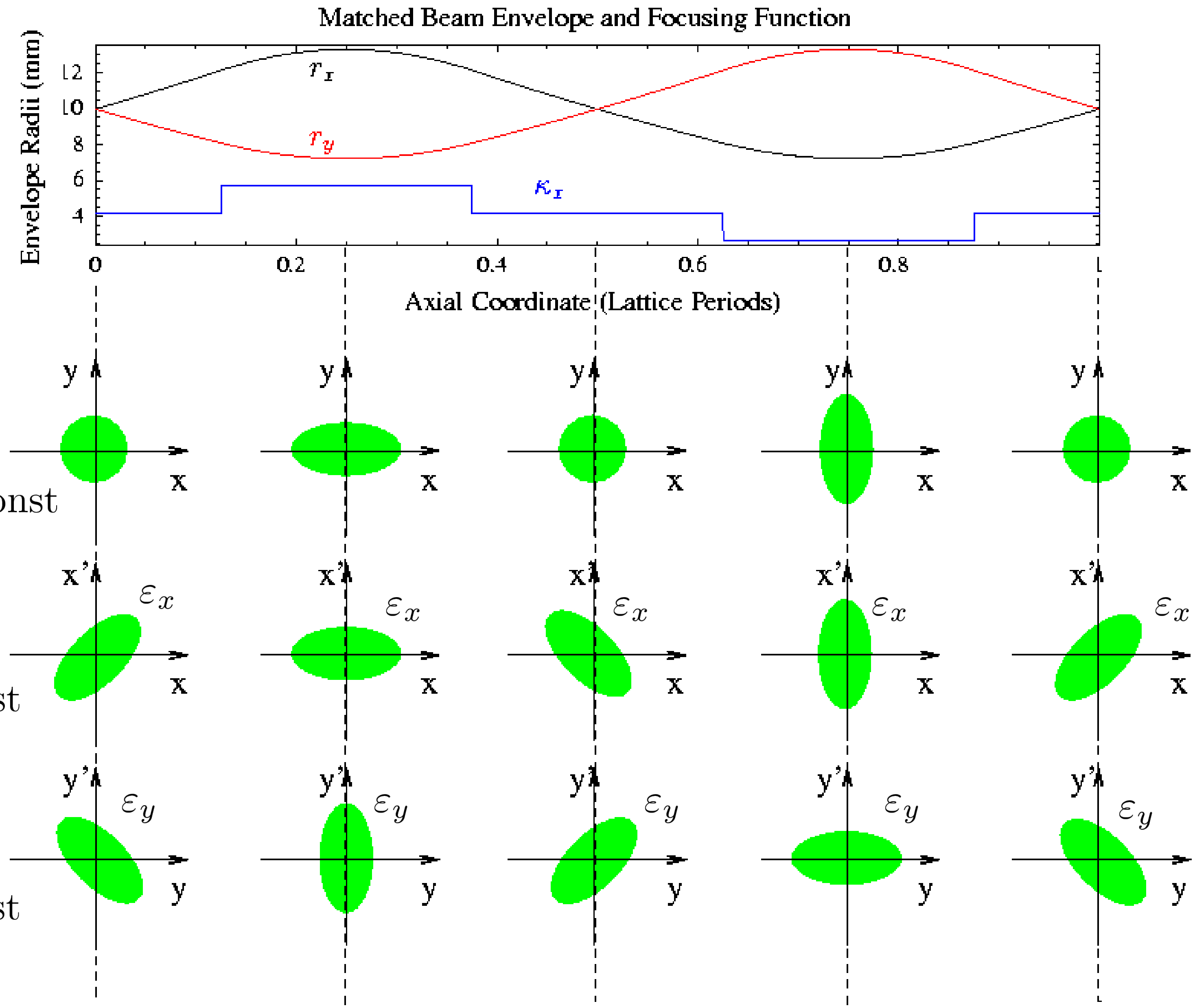
Typical Matched vs Mismatched solution for FODO channel:



The matched beam is the most radially compact solution to the envelope equations rendering it highly important for beam transport

- Matching uses optics most efficiently to maintain radial beam confinement

Skip: Symmetries of a matched beam are interpreted in terms of a local rms equivalent KV beam and moments/projections of the KV distribution



Projection

x-y

area: $\pi r_x r_y \neq \text{const}$

x-x'

area: $\pi \varepsilon_x = \text{const}$
(CS Invariant)

y-y'

area: $\pi \varepsilon_y = \text{const}$
(CS Invariant)

S3: Characteristic Transverse Particle Orbits Including Space-Charge

Continuous focusing, axisymmetric beam

$$\boxed{\begin{aligned} &\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \text{const} \\ &\varepsilon_x = \varepsilon_y \equiv \varepsilon \\ &r_x = r_y \equiv r_b \end{aligned}}$$

Undepressed betatron wavenumber

Envelope equation

$$r_x'' + \kappa_x r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} = 0$$

$$r_y'' + \kappa_y r_y - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_y^2}{r_y^3} = 0$$

reduces to:

$$r_b'' + k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon^2}{r_b^3} = 0$$

with matched ($r_b' = 0$) solution to the quadratic in r_b^2 envelope equation

$$\boxed{r_b = \left(\frac{Q + \sqrt{4k_{\beta 0}^2 \varepsilon^2 + Q^2}}{2k_{\beta 0}^2} \right)^{1/2} = \text{const}}$$

Similarly, neglecting images, particle equations of motion within the beam are:

$$x'' + \kappa_x x = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial x}\phi$$

$$y'' + \kappa_y y = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial y}\phi$$

Uniform Elliptical Beam $\Longrightarrow$

$$x'' + \left\{\kappa_x - \frac{2Q}{[r_x + r_y]r_x}\right\}x = 0$$

$$y'' + \left\{\kappa_y - \frac{2Q}{[r_x + r_y]r_y}\right\}y = 0$$

reduce for a continuously focused axisymmetric beam to

$$\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \text{const}$$

$$\mathrm{r}_x = r_y \equiv r_b = \text{const}$$

$$\Longrightarrow \quad x'' + \left\{k_{\beta 0}^2 - \frac{Q}{r_b^2}\right\}x = 0$$

$$y'' + \left\{k_{\beta 0}^2 - \frac{Q}{r_b^2}\right\}y = 0$$

$$\boxed{\begin{aligned} x'' + k_\beta^2 x &= 0 \\ y'' + k_\beta^2 y &= 0 \end{aligned}}$$

$$k_\beta \equiv \sqrt{k_{\beta 0}^2 - \frac{Q}{r_b^2}} = \text{const}$$

Depressed betatron wavenumber

The linear equations of motion for particles within the beam

$$\boxed{\begin{aligned} x'' + k_\beta^2 x &= 0 \\ y'' + k_\beta^2 y &= 0 \end{aligned}}$$

have simple harmonic oscillator solutions

$$\boxed{\begin{aligned} x(s) &= x_i \cos[k_\beta(s - s_i)] + \frac{x_i'}{k_\beta} \sin[k_\beta(s - s_i)] \\ y(s) &= y_i \cos[k_\beta(s - s_i)] + \frac{y_i'}{k_\beta} \sin[k_\beta(s - s_i)] \end{aligned}}$$

$x_i = \text{Inital } x\text{-Coordinate}$ $\quad$ $y_i = \text{Inital } y\text{-Coordinate}$

$x_i' = \text{Inital } x\text{-Angle}$ $\quad$ $y_i' = \text{Inital } y\text{-Angle}$

$$s = s_i = \text{Initial } s\text{-Coordinate}$$

Space-charge tune depression (rate of phase advance same everywhere, lattice period arbitrary)

$$k_{\beta 0} = \text{Wavenumber focus, no space-charge} \qquad k_{\beta 0} \simeq \frac{\sigma_0}{L_p}$$

$$k_\beta = \sqrt{k_{\beta 0}^2 - \frac{Q}{r_b^2}} = \text{Wavenumber with space-charge} \qquad k_{\beta 0} \simeq \frac{\sigma}{L_p}$$

$$\boxed{\begin{array}{c}\text{Space-Charge}\\ \text{Tune Depression}\end{array} \equiv \frac{\sigma}{\sigma_0} = \frac{k_\beta}{k_{\beta 0}} = \left(1 - \frac{Q}{k_{\beta 0}^2 r_b^2}\right)^{1/2}}$$

Space-Charge Limit (Cold Beam)

Single Particle Dynamics (Warm Beam)

$$0 \quad \leq \quad \frac{\sigma}{\sigma_0} \quad \leq \quad 1$$

$\varepsilon \rightarrow 0$ $\qquad$ $Q \rightarrow 0$

envelope equation

$$k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon^2}{r_b^3} = 0$$

envelope equation

$$\Rightarrow r_b = \sqrt{Q}/k_{\beta 0}$$

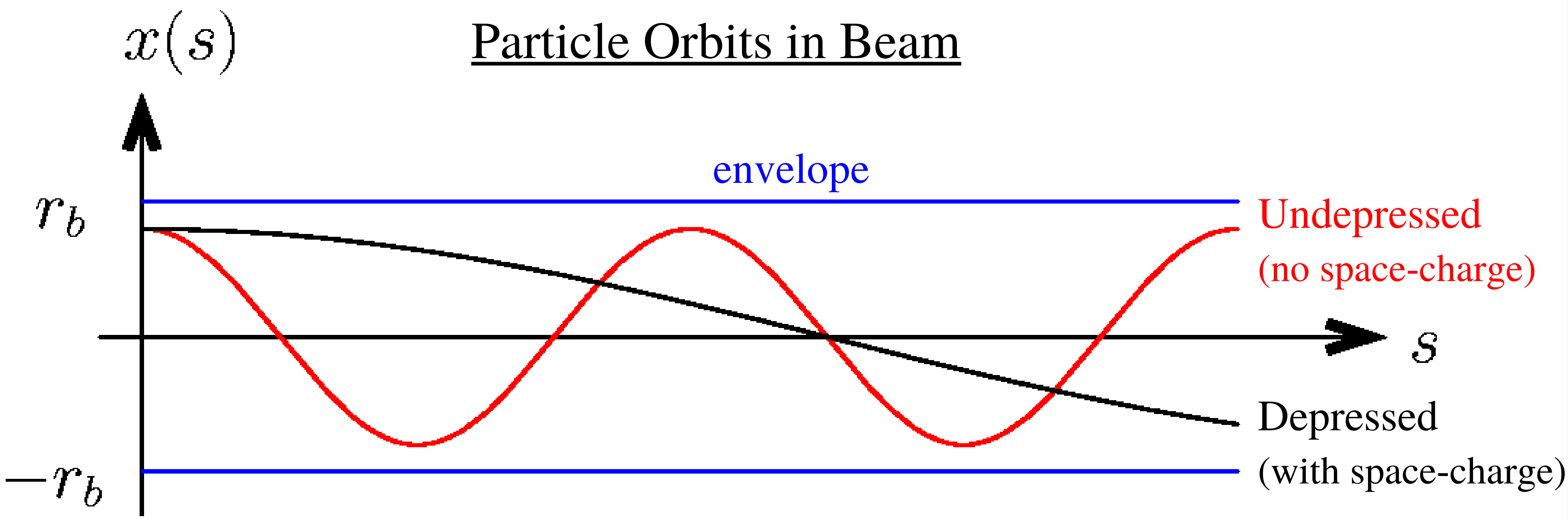
envelope equation

$$\Rightarrow r_b = \sqrt{\varepsilon/k_{\beta 0}}$$

Continuous Focusing KV Equilibrium –
Undepressed and depressed particle orbits in the *x*-plane

$$k_\beta = \frac{\sigma}{\sigma_0} k_{\beta 0} \qquad \frac{\sigma}{\sigma_0} = 0.2 \qquad y = 0 = y'$$

Particle Orbits in Beam



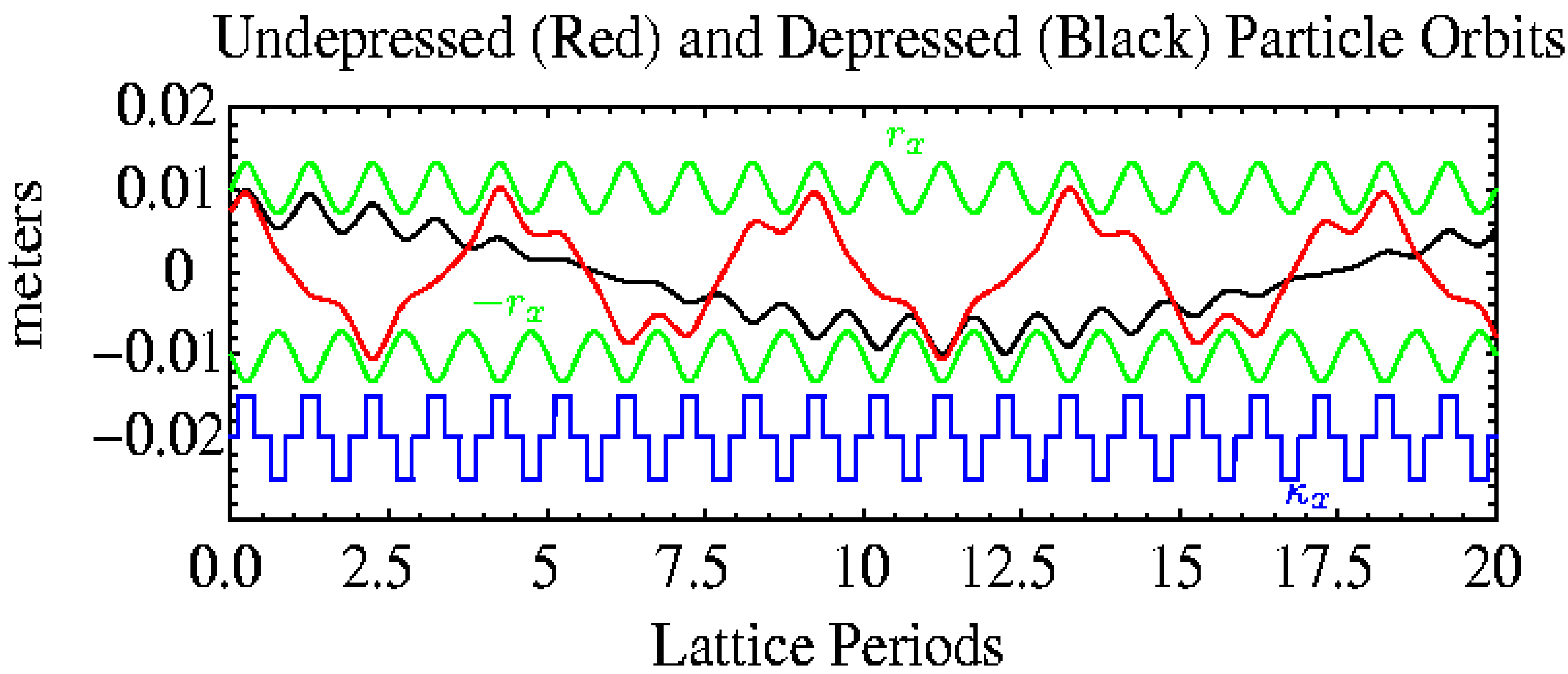
Much simpler in details than the periodic focusing case, but qualitatively similar in that space-charge "depresses" the rate of particle phase advance.

- At full space-charge depression oscillations have zero phase advance

Depressed particle x-plane orbits within a matched periodic FODO quadrupole channel: No details but illustrate intricate nature of orbits

x-plane orbit:
$$y = 0 = y'$$

Q Tuned for:
$$\sigma_0 = 80^\circ$$
$$\frac{\sigma}{\sigma_0} = 0.2$$



S4: Perspective

Beam Space Charge is intrinsically defocusing and results in a collective response

- Acts continuously and most important at low energy (near injector)
  - Longitudinal pulse compression can make important at higher energy
- Intricate physics: similar to classical plasma physics but complex due to "equilibrium" self-fields
- Collective response can have rich waves spectrum, and some waves can be destabilizing or generate beam halo

Examples of collective wave relaxation of an intense beam:

Continuous Focus Waves:

https://people.nscl.msu.edu/~lund/msu/phy862_2018/tks_relax_cf.mpg

AG Quadrupole Focus Waves:

https://people.nscl.msu.edu/~lund/msu/phy862_2018/tks_relax_ag.mpg

Only most basic introduction here

- Longitudinal space-charge effects also: dropped for lack of time
  - Proximity of conducting pipe strongly alters longitudinal self-field of long beam
- Due to difficult nature of analysis self-consistent simulations central to topic
  - Particle in Cell (PIC) codes common
- If interested, consider taking US Particle Accelerator School (USPAS) courses on topic after graduate accelerator physics

Corrections and suggestions for improvements welcome!

These notes will be corrected and expanded for reference and for use in future editions of the course. Contact:

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Please provide corrections with respect to the present archived version at:

https://people.nscl.msu.edu/~lund/msu/phy862_2019/

Redistributions of class material welcome. Please do not remove credits.

References: For more information see:

USPAS "Beam Physics with Intense Space-Charge" course notes with updates, corrections, and supplemental materials:

https://people.nscl.msu.edu/~lund/uspas/bpisc_2017

Basic introduction on many of the topics covered:

M. Reiser, *Theory and Design of Charged Particle Beams*, Wiley (1994, revised edition 2008)