SPACE CHARGE DOMINATED BEAM TRANSPORT

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Abstract
We consider beam transport systems where space charge forces are comparable in strength with the external focusing force. Space charge then plays an important role for beam transmission and emittance growth. We use the envelope model for matching and the generalized field energy equations to study emittance growth. Analytic results are compared with numerical simulation.

1 INTRODUCTION

In high-current linear accelerators and in transport systems for protons or heavier ions the repulsive force due to the space charge carried by the beam itself plays an important role for the design of the focusing system and for conservation of beam emittance. In proton or heavy ion linear accelerators the actual space charge bottle-neck is at injection, where the ion is slow and correspondingly the space charge density and resulting forces large. Much can be learnt from transport experiments [1, 2, 3], which have been performed to study emittance growth under stationary conditions, i.e. no acceleration. Interest in these transport experiments was largely stimulated by the idea of using heavy ions for inertial confinement fusion [4], which requires transport of large currents over long distances. We emphasize, however, that the role of space charge for emittance growth is equally important for high-current proton or heavy ion linacs. One way of discussing emittance growth is by relating it to the electrostatic energy of the space charge distribution ("nonlinear field energy"). This concept was originally derived in Refs. [5, 6],

for 2-D beams. The analytical theory can be generalized to the 3-D case [7], but we confine this lecture to the 2-D problem of beam transport of long beams, ignoring the longitudinal degree of freedom.

2 BASIC PROPERTIES

2.1 Incoherent Effect of Space Charge

For comparison we recall that in circular accelerators space charge is always a relatively weak perturbation described by the betatron tune shift $\Delta\nu$

$$\frac{d^2x}{d\Theta^2} = -\left(\nu_0 - \Delta\nu\right)^2 x = -\nu^2 x \tag{1}$$

where ν_0 is the betatron tune in the absence of space charge and $\Delta\nu < \frac{1}{4} << \nu_0$ in order to use a gap free of machine resonances. The corresponding slight increase in betatron wavelength is contrasted by a large effect in high-current linear accelerators, where one tries to compensate the external focusing force (given by ν_0^2) as much as possible by space charge, hence $\nu^2 < \nu_0^2$ (Fig. 1).

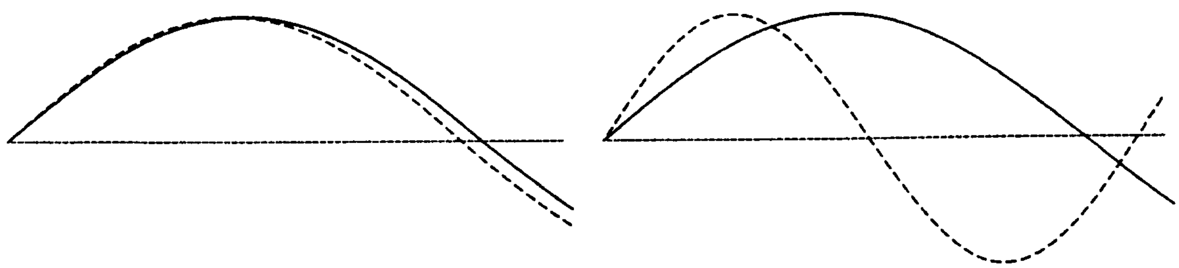


Figure 1: Space charge effect on betatron oscillation in circular accelerators (left) and high-current linear accelerators (right).

2.2 Coherent Effects

In practice the ideal space charge limit $\nu = 0$ with an exact balance between the applied focusing and the defocusing force due to space charge can be approached only to a limited extent. With decreasing ν/ν_0 the beam is increasingly dominated by collective behaviour, which can lead to emittance growth. Coherent modes of oscillation have been studied analytically for the highly specialized Kapchinskij-Vladimirskij distribution [8]. For more realistic beam models one depends on computer simulation, which shows various modes of emittance growth due to such coherent oscillations (reviewed in Ref. [9]). It will be shown below that in a periodic focusing system coherent modes of oscillation can be in resonance with the focusing period. "Harmful" structure resonances can be largely avoided by choosing sufficiently small values for σ_o.

2.3 Space Charge Limited Current

Following Maschke [10] the relationship between beam current and σ/σ_0 for a periodic quadrupole channel can be expressed in the following way [11]:

$$I = 3.66 \cdot 10^6 (\frac{A}{Z})^{1/3} B_0^{2/3} (\beta\gamma)^{7/3} \varepsilon^{2/3} F(\sigma/\sigma_0) \quad (2)$$

with B_0 the quadrupole pole-tip field, ε the transverse emittance and $F(\sigma/\sigma_0)$ a factor, which depends only weakly on the type of focusing (see Fig. 2 for a symmetric quadrupole channel). Equation (2) is the condition that the beam is matched to the transport channel. For small σ/σ_0 the dependence of beam current on σ for a given channel can be seen more directly from the approximately valid scaling [12]

$$I \sim a^2 \sim \frac{\varepsilon}{\sigma} \quad (3)$$

where a is the channel radius. Knowledge of the minimum σ for stable transport is thus essential for optimum current transmission. This will be studied in more detail further below.

3 MATCHING WITH UNIFORM SPACE CHARGE

The defocusing effect of space charge has to be compensated by an appropriate change of the α- and β-functions at the entrance to a transport channel. For large space charge it is often necessary to increase the quadrupole strength to avoid hitting the aper-

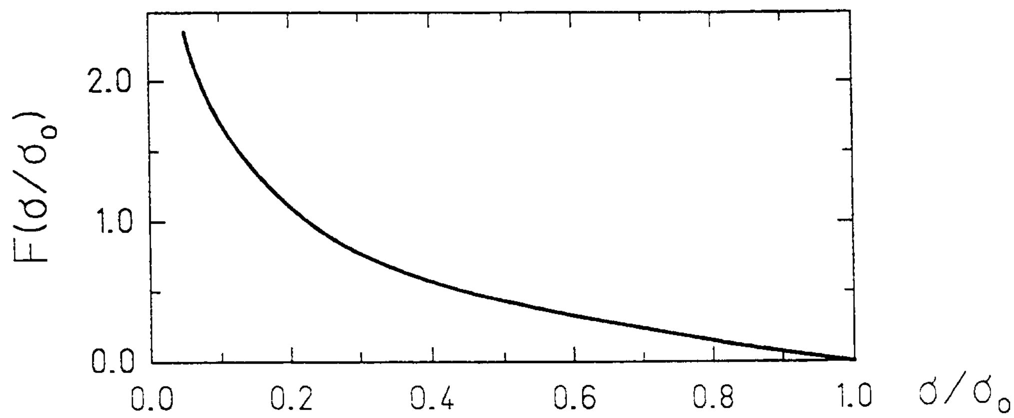


Figure 2: Current transmission factor as function of σ/σ_0.

ture. For uniform space charge distribution within an elliptic boundary the repulsive Coulomb force varies linearly with distance from the beam center:

$$E_x = \frac{I}{\pi\varepsilon_0 v_0} \, \frac{x}{a(a+b)} \tag{4}$$

and

$$E_y = \frac{I}{\pi\varepsilon_0 v_0} \, \frac{y}{b(a+b)} \tag{5}$$

with $v_0 = \beta c$ and a, b the transverse semi-axi (i.e. envelopes). As is shown in the appendix, following the derivation by Sacherer, [13] one can then describe the beam envelopes by the equations

$$\frac{d^2}{ds^2}a + k_x(s)a - \frac{\varepsilon_x^2}{a^3} - \frac{qI}{\pi\varepsilon_0 m\gamma^3 v_0^2(a+b)} = 0 \tag{6}$$

and similarly

$$\frac{d^2}{ds^2}b + k_z(s)b - \frac{\varepsilon_z^2}{b^3} - \frac{qI}{\pi\varepsilon_0 m\gamma^3 v_0^2(a+b)} = 0, \tag{7}$$

with $\varepsilon_x, \varepsilon_z$ the emittances and I the current. These equations are the basis for matching of a beam to a transport channel in the presence of space charge.

As an example we consider a symmetric FODO transport channel with the periodic cell 4 m long and a beam of protons at 100 MeV with $\varepsilon = 10^{-6}\pi$m-rad. A matched envelope for vanishing space charge ($I = 0$) is shown in Fig. 3 (top), where the upper trace is the horizontal, and the lower trace the vertical envelope. The trajectory of the test particle shown gives a full oscillation over six periodic cells, i.e. $\sigma_0 = 60^0$. We also show the increased envelope and reduced betatron phase advance σ for large currents, but focusing gradients unchanged. Results are summarized in Fig. 4, along with the increase of β-function values (here at the center of a drift section) for decreasing σ.

For an aperture limited beam transport it is necessary to increase the focusing gradients. In Fig. 5 we have used the same currents as in Fig. 3, but increased the

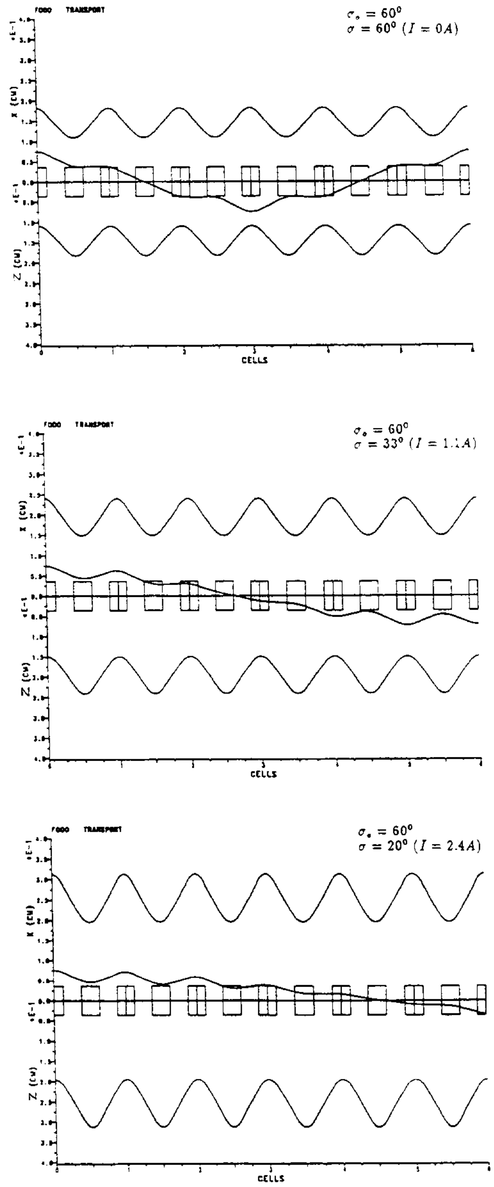


Figure 3: Space charge effect on matched envelopes in symmetric FODO channel with $\sigma_0 = 60^0$.

quadrupole gradients (shown by the height of square boxes) so that $\sigma = 60^0$ is maintained for the depressed tune. As a consequence the average envelope has about the same size as in the zero-current case of Fig. 3, whereas the excursions of the envelope increase with current.

4 EMITTANCE AND FIELD ENERGY

4.1 General Principle

We are interested to understand under what conditions and to what extent the emittance can grow. Our goal is to predict emittance growth from general principles - as much as possible - without actually computing the dynamical process responsible for it. Here we present an approach, which allows a qualitative discussion of the various sources of emittance growth, as well as quantitative estimates for the expected growth in special cases. The general idea of this approach is to consider emittance growth as an increase

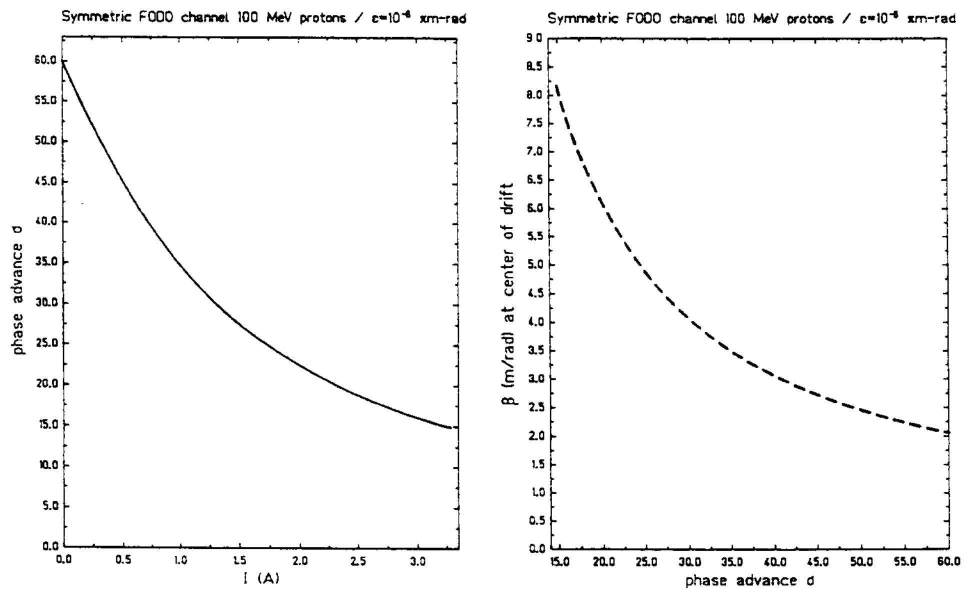


Figure 4: Phase advance σ as function of I and β-function as function of σ for fixed $\sigma_0 = 60^0$.

of thermal energy, which must be taken from the overall available beam energy. If we succeed in estimating the available source of energy we can also estimate the maximum possible emittance growth.

We assume the total energy of the beam can be written in the following way

$$E_{total} = E_0 + E_{th,x} + E_{th,z} + E_{ext} + E_{field} \tag{8}$$

where

x, z	= transverse coordinates
E_0	= forward kinetic energy
$E_{th;x,z}$	= thermal energy in x, z (equivalent to emittance)
E_{ext}	= potential energy due to externally applied focusing force
E_{field}	= energy of space charge induced electric field

A growth of thermal energy (in x or z) and thus emittance is possible at the expense of either E_0 or E_{field}. Due to the fact that the beam is always at a minimum of the external potential (in time average), E_{ext} is not a source of energy available for emittance growth and we expect E_{ext} also to increase during growth of the emittance. In addition, it is possible that thermal energy is transferred between the transverse plane (x, z) and the longitudinal direction (y), if there is a significant initial imbalance. This important subject ("equipartitioning") is treated in Ref. [14], with an analytical model in Ref. [15]. In all cases E_{field} plays a crucial role. It should be noted here that in a constant focusing system E_{th} + E_{ext} + E_{field} is constant, hence we have no coupling with E_0. In periodic focusing, on the other hand, such a coupling can be significant as will be shown below. This subject has been treated in detail in Ref. [8].

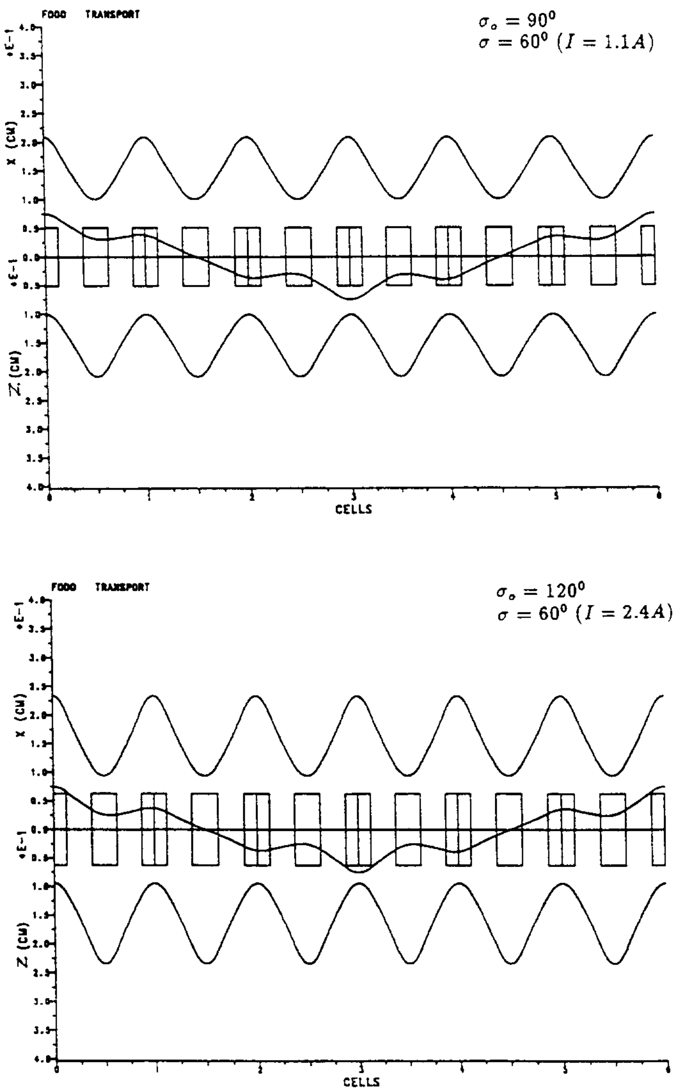


Figure 5: Increased focusing gradients keeping constant $\sigma = 60^0$, which results in roughly constant average envelope.

4.2 Generalized Emittance Equation

It can be shown that the emittances $\varepsilon_x, \varepsilon_z$ for the two transverse planes x, z are related to the field energy by the following equation, valid for constant or periodic focusing (the derivation is given in Appendix A; see also Refs. [5, 6, 7]:

$$\frac{1}{\overline{x^2}} \frac{d}{ds} \varepsilon_x^2 + \frac{1}{\overline{z^2}} \frac{d}{ds} \varepsilon_z^2 = -4K \frac{d}{ds} \frac{W - W_u}{w_0} \quad (9)$$

with

$$K = \frac{Nq^2}{2\pi\varepsilon_0 m\gamma^3 v_0^2} \quad (generalized\ \ perveance)$$

$\overline{x^2}, \overline{z^2}$ mean squares of x, z

W actual field energy
W_u field energy of equivalent uniform density beam
w_0 normalization constant (= field energy inside actual beam boundary of a uniform beam)

The significance of this equation is that the change of emittance is related to a change of the "nonlinear field energy" $W - W_u$, i.e. the excess field energy due to the non-uniformity of the density. A uniform density beam — even with large mismatch — therefore has constant emittance.

4.3 Minimum Field Energy of Uniform Density Beam

A practically important property of the field energy is that it adopts its minimum for a uniform density beam. This can be shown by defining a variational expression

$$S \equiv W + \alpha_1 x^2 + \alpha_2 z^2 \tag{10}$$

with α_1, α_2 Lagrangian multipliers to keep the r.m.s. size constant. For a minimum we require the variation of S to vanish, hence

$$\delta S = \int\int \left[\varepsilon_0 E \delta E + N^{-1}\left(\alpha_1 x^2 + \alpha_2 z^2\right)\delta n\right] dx\ dz. \tag{11}$$

The density perturbation is defined by an arbitrary displacement

$$\delta n(x,z) = \nabla n \cdot \delta \mathbf{x}, \tag{12}$$

thus we obtain

$$\delta S = \int\int \left[\phi + N^{-1}\left(\alpha_1 x^2 + \alpha_2 z^2\right)\right](\nabla n \delta \mathbf{x})\, dx dz = 0 \tag{13}$$

requiring either $\phi = -N^{-1}(\alpha_1 x^2 + \alpha_2 z^2)$ (interior of beam, hence uniformly charged ellipsoid) or $n = const. = 0$ (exterior).

4.4 Field Energy of Different Beam Models

Calculation of the field energy for a parabolic density profile

$$n = \frac{2N}{\pi ab}\left(1 - \frac{x^2}{a^2} - \frac{z^2}{b^2}\right) \tag{14}$$

leads to the following result

$$W = \frac{N^2 q^2}{16\pi\varepsilon_0}\left(\frac{11}{6} - 4\ \ln\sqrt{6} + 4\ \ln\frac{2R}{\tilde{x} + \tilde{z}}\right) \tag{15}$$

with the r.m.s. envelope $\tilde{x} = a/\sqrt{6}$ and $\tilde{z} = b/\sqrt{6}$. Due to the minimum field energy property of a uniform beam of the same r.m.s. size it is convenient to calculate the difference energy in normalized units:

$$\frac{W - W_u}{w_0} = \frac{5}{6} - 4\ \ln\frac{\sqrt{6}}{2} \simeq 0.0224. \tag{16}$$

This normalized "nonlinear field energy" hence is always positive according to the previous section. For a Gaussian density profile truncated at 4 standard deviations one obtains the considerably larger value of 0.154.

4.5 Stationary Distributions and Debye Shielding

It must be noted that a Gaussian phase space distribution is consistent with a Gaussian density profile only in the low-current limit. From computer simulation one finds that practically any phase distribution functions leads to uniform density in the limit of high current. In Fig. 6 this is shown for a Gaussian distribution.

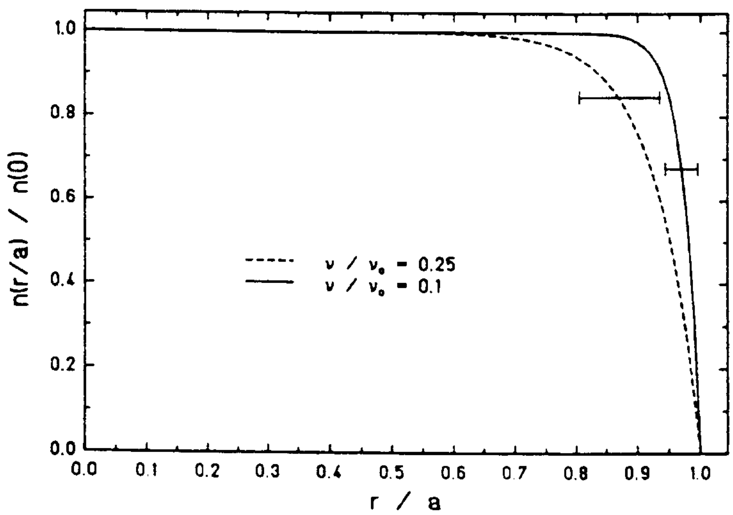


Figure 6: Density profiles for Gaussian phase space distribution near space charge limit. Bars indicate λ_D according to Eq. (20).

This is readily shown for a Gaussian distribution ($r^2 = x^2 + z^2$)

$$f \sim \exp\left[-\left(\frac{x'^2 + z'^2}{2} + \frac{k}{2}\,r^2 + \frac{q}{m\gamma^3 v_0^2}\phi\right)/\mu\right] \tag{17}$$

which, after insertion into Poisson's equation and expansion of the exponential yields

$$n \simeq n_0\left[1 - \left(\frac{a}{r}\right)^{1/2}\exp\left((r-a)/\lambda_D\right)\right] \tag{18}$$

provided that the total potential in Eq. (17) is small as compared with the average transverse energy μ. This condition can be expressed in terms of the Debye-length λ_D, i.e.

$$\frac{\lambda_D}{a} << 1 \tag{19}$$

where λ_D is given by

$$\lambda_D^2 = \frac{\mu}{\omega_p} \simeq 2\frac{\overline{x'^2}}{\omega_p^2} \simeq \frac{v_{th}^2}{\omega_p^2} \tag{20}$$

We can also express λ_D/a in terms of the tune depression

$$\frac{\lambda_D}{a} \simeq \frac{1}{\sqrt{8}} \frac{\nu}{\nu_0} \tag{21}$$

where ν/ν_0 is replaced by σ/σ_0 for periodic focusing.

The physical meaning of the Debye-length is the one familiar in plasma physics: a high-current beam shields the external focusing force from its interior by building up a uniform density with a space charge force that cancels the focusing force. The shielding is ineffective in the boundary layer of a thickness given by λ_D (Fig. 6). Hence, for small ν/ν_o there is practically no restoring force in the interior of the beam and particles are "reflected" by the uncompensated force in the boundary layer.

5 APPLICATION TO EMITTANCE GROWTH

Using Eq. (9) we can now discuss the possible sources for emittance growth. The relationship between emittance and thermal rsp. kinetic energy suggests the analogy with a ball in a potential trough. The following situations can be envisaged:

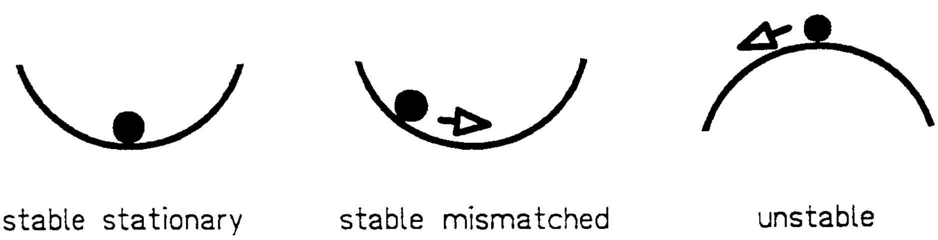


Figure 7: Analogy with ball in a potential trough

5.1 Stable Stationary Beam

This requires that the emittances ε_x and ε_z are conserved along the channel and the envelopes in x and z vary periodically. According to Eq. (9) a necessary condition for this is that $W - W_u$ is a constant. This can be satisfied if the beam density profile remains self-similar. Such an observation has been made in numerical simulation; an analytical proof for beams with non-uniform density and periodic focusing is given in Ref. [16].

5.2 Mismatched Beam

An initial excess field energy is transferred into emittance according to Eq. (9). For a round beam and constant focusing we can integrate Eq. (9) and find

$$\frac{\Delta\varepsilon^2}{\varepsilon^2} = -\frac{1}{2}\left(\frac{\nu_0^2}{\nu^2} - 1\right)\Delta U \tag{22}$$

with $U \equiv (W - W_u)/w_0$ the normalized nonlinear field energy.

As an example we inject a beam with Gaussian profile into a channel with $\nu_0/\nu = 6$. Since we know from equation (14) that close to the space charge limit a stationary solution has nearly uniform density, nearly all of the excess nonlinear field energy (0.15 in normalized units) is found as emittance growth during less than a betatron oscillation. Hence we can estimate the predicted emittance growth by ignoring the final nonlinear field energy and thus obtain

$$\frac{\Delta\varepsilon^2}{\varepsilon^2} \leq 2.6 \tag{23}$$

which is equivalent to an emittance growth of 90 % and agrees very well with the result of simulation. This is shown in Fig. 8a for a multiparticle computer simulation in a periodic solenoid channel. We observe that the mismatch emittance growth formula applies equally well to periodic focusing also - growth factors found in simulation are practically identical with those in constant focusing. It is only the ratio ν_0/ν rsp. σ_0/σ, which is important.

A different result is obtained if a strictly uniform beam is injected, hence we start at the minimum field energy. Such a beam has the tendency of smoothing its sharp boundary, which requires a small amount of field energy leading to a small decrease in r.m.s. emittance. This is again confirmed by simulation (Fig. 8b). We observe that such an r.m.s. emittance decrease is not violating Liouville, which still applies to the four-dimensional phase space. We thus conclude that injection of as uniform a density as possible minimizes emittance growth of high-current beams.

In this context we also need to mention the possibility of an initial envelope mismatch, which might result from an improper entrance matching. The associated "mismatch energy" can be translated into emittance growth if the beam density is nonuniform. This case is considered in more detail in Ref. [17].

5.3 Coherent Instabilities

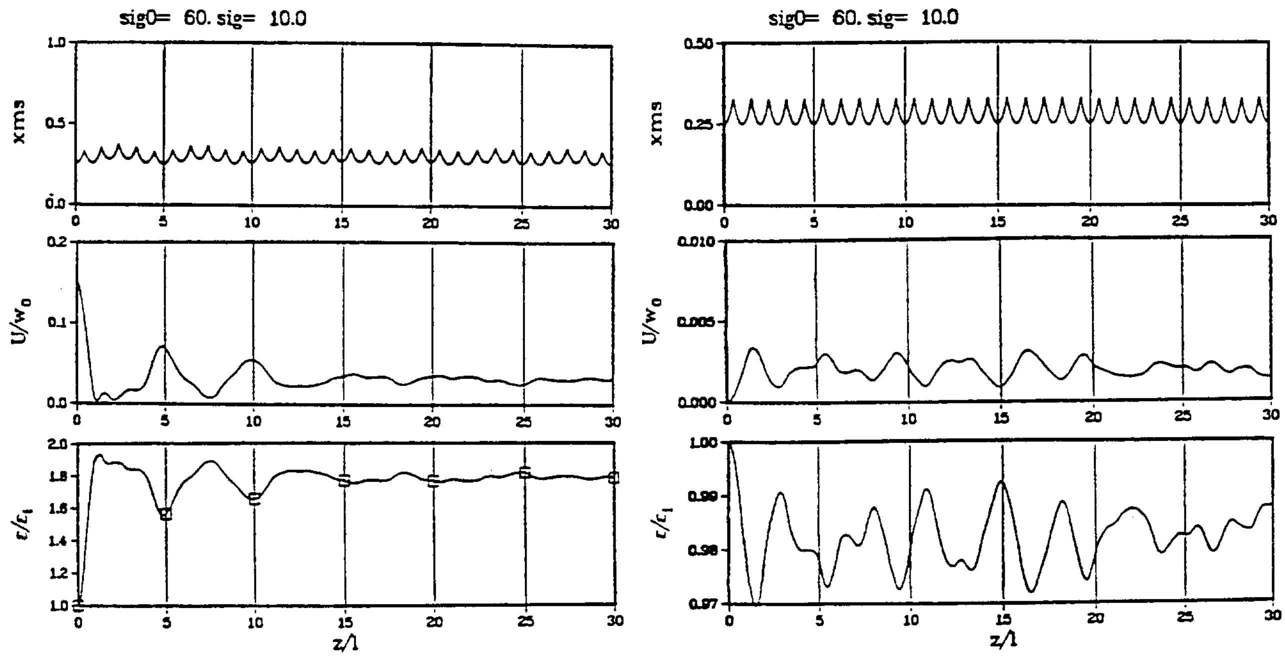
In linear beam transport, coherent instabilities occur due to a local interaction of an ensemble of particles via the space charge force. This is in contrast with beams in circular accelerators, where coupling to the surrounding structure or to other bunches plays a dominant role.

In the present framework coherent instabilities can be discussed qualitatively by writing Eq. (9) for a round beam in periodic solenoidal focusing ($k_x = k_z$).

$$\frac{d}{ds}\varepsilon^2 = -2K\,\overline{x^2(s)}\frac{d}{ds}\,\frac{W - W_u}{w_0}. \tag{24}$$

We assume a matched envelope, hence $\overline{x^2}$ oscillates with the focusing periodicity. Furthermore, we assume a small coherent oscillation of the beam, such that $W - W_u$ oscillates with the period of oscillation of the particular mode. We now anticipate a steadily growing ε^2, if $\overline{x^2}$ and $W - W_u$ oscillate with the same frequency (phase shifted by $\sim 90^0$). In this case the r.h.s. of Eq. (24) has a non-oscillatory part, which adds up to a finite emittance growth until the resonance condition is lost by detuning. The resulting "structure resonance" is thus a coupling between the zero-order oscillatory envelope and a non-uniform coherent mode of oscillation as illustrated by the example in Fig. 9 of a round beam in periodic focusing with $\sigma_0 = 120^0$ and $\sigma = 10^0$.

A direct calculation of the emittance growth is not possible from Eq. (24); this would require to determine the actual time dependence of $W - W_u$, which is not possible analyt-



(a) (b)

Figure 8: Squared envelope (top), nonlinear field energy U/w_0 (center) and emittance growth factors (bottom) versus number of focusing periods obtained from multiparticle computer simulation in a periodic solenoid channel with $\sigma_0 = 60^0, \sigma = 10^0$. (a) Gaussian density profile, (b) Semi-Gaussian: uniform density profile and Gaussian velocity distribution.

ically. For illustration of the principle of a structure resonance we can, however, derive a scaling expression for the emittance growth by making a simple harmonic approximation

$$\overline{x^2(s)} \simeq \hat{x}^2 + \delta x^2 \sin(\omega s) \tag{25}$$

$$U \simeq \hat{U} + \delta U \cos(\omega s) \tag{26}$$

where $\omega = 2\pi/L$ and L the focusing period. We then find from Eq. (24)

$$\frac{d}{ds}\varepsilon^2 \simeq 2K\delta x^2 \omega \; \delta U \sin^2(\omega s) \tag{27}$$

In smooth approximation K follows from the envelope equation (with ε_i the matched initial emittance)

$$K =_4^1 \frac{\varepsilon_i^2}{\hat{x}^2}\left(\frac{\sigma_0^2}{\sigma^2} - 1\right). \tag{28}$$

We obtain

$$\frac{\varepsilon}{\varepsilon i} \leq \left[1 + \frac{\pi}{2}\left(\frac{\sigma_0^2}{\sigma^2} - 1\right)\frac{\delta x^2}{\hat{x}^2}\delta U \frac{s}{L}\right]^{1/2}, \tag{29}$$

where we have used the approximation $\int \sin^2(\omega s) ds \approx \frac{S}{2}$.

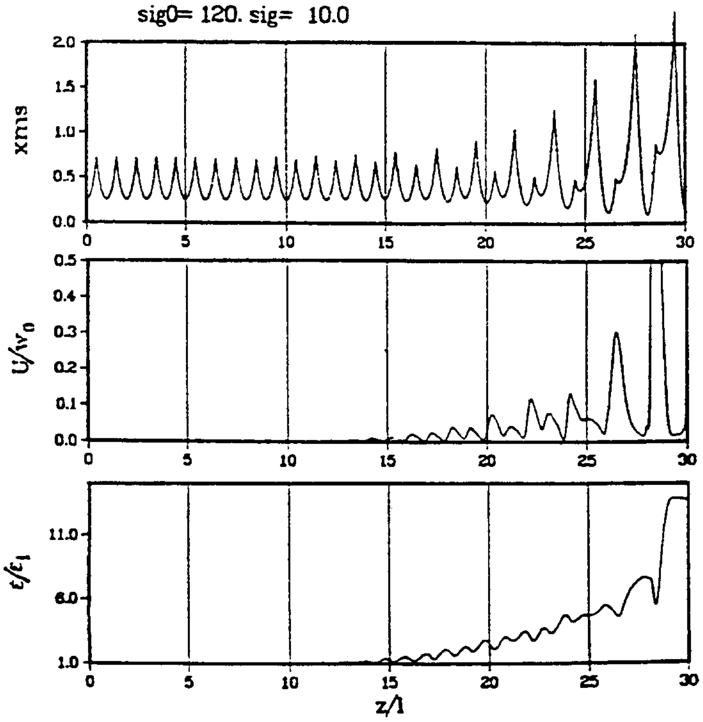


Figure 9: Structure resonance in computer simulation of initially uniform beam (KV-distribution) in periodically interrupted solenoid focusing.

From this expression we conclude that in an advanced stage the emittance rise is weaker than linear in time. It is enhanced by small σ and large envelope excursions as in strong focusing. In the example of Fig. 9 the excited mode has an octupole type field perturbation, which grows from noise on the initially uniform beam (KV-distribution) and saturates between focusing period 15 and 20. In this region the average normalized nonlinear field energy $\delta \bar{U}$ is estimated as 0.02 and Eq. (29) yields $\varepsilon/\varepsilon_0 \simeq 3$ which agrees reasonably well with the numerical result. After focusing period 20 the situation becomes very nonlinear: a different mode with half the oscillation frequency starts growing and leads to further strong emittance growth.

These particular modes can be avoided if $\sigma_0 \leq 90^0$, in which case they are out of resonance. As was shown in Ref. [8] another type of mode can appear for $\sigma_0 = 90^0$ and $\sigma \simeq 45^0$ (sextupole type field perturbation). This mode, however, was found to be practically negligible in its effect on emittance growth. This result was also confirmed by experiment [1].

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A APPENDIX

A.1 Derivation of R.M.S. Envelope Equation

In order to derive analytically a relationship between emittances and field energy we consider x and z as transverse coordinates and start from the single-particle equation of motion in x (similar for z)

$$x'' + k_x(s)x - \frac{q}{m\gamma^3 v_0^2} E_x(x, z, s) = 0 \tag{A.1}$$

where k_x describes the periodically varying focusing force and $\mathbf{E}$ follows from Poisson's equation

$$\nabla \cdot \mathbf{E} = \frac{q}{\varepsilon_0}\, n(x, z, s) \tag{A.2}$$

and n is the density obtained by projecting a 4-D phase space distribution into real space:

$$n = \int\int f\,(x, z, x', z', s)\, dx'\; dz' \tag{A.3}$$

The distribution function f is subject to Vlasov's equation ($v_0 \equiv dx/ds$).

$$\frac{\partial f}{\partial s} + (\mathbf{x}' \cdot \nabla)\; f - \left(\mathbf{k} - \frac{q}{m\gamma^3 v_0^2}\mathbf{E}\right) \cdot \nabla'_x f = 0 \tag{A.4}$$

Solution of equations (A.1)-(A.4) is fraught with the usual difficulties in solving a partial integro-differential equation. Keeping in mind that the r.m.s. emittance is defined through second-order moments of the distribution function, we follow the procedure introduced by Sacherer [13] and convert Eq. (A.4) into ordinary differential equations involving second-order moments only. We thus define the moments, where we assume that N is the total number of particles per unit length of beam, according to

$$\overline{x^2} = N^1 \int \ldots \int x^2 f\; dx \ldots dz', \tag{A.5}$$

similar with x'^2, xx' and likewise in z. The r.m.s. emittance is then defined as (note that some authors drop the factor 4)

$$\varepsilon_x \equiv 4\left(\overline{x^2 x'^2}\; - \;\overline{xx'}^2\right)^{1/2} \tag{A.6}$$

and similar in z. We then obtain from Eq. (A.4) by multiplying it with x^2 and integrating over all phase space

$$\frac{d}{ds}\overline{x^2} - 2\overline{xx'} = 0 \tag{A.7}$$

$$\frac{d}{ds}\overline{xx'}\; - \;\overline{x'^2} + k_x\overline{x^2} - \frac{q}{m\gamma^3 v_0^2}\overline{xE_x} = 0 \tag{A.8}$$

$$\frac{d}{ds}\overline{x'^2} + 2k_x\overline{xx'} - \frac{2q}{m\gamma^3 v_0^2}\;\overline{x'E_x} = 0 \tag{A.9}$$

By applying Sacherer's procedure we readily obtain the equation

$$\frac{d^2}{ds^2}\tilde{x} + k_x(s)\tilde{x} - \frac{\varepsilon_x^2(s)}{16\tilde{x}^3} - \frac{q}{m\gamma^3 v_0^2}\frac{\overline{xE_x}}{\tilde{x}} = 0 \tag{A.10}$$

(similar in z) with $\tilde{x} \equiv \overline{x^2}^{1/2}$ the r.m.s. envelope. This requires, however, knowledge of ε_x to be of real use. Hence, for constant ε_x, a straightforward integration of the r.m.s. envelope equation is possible, if $\overline{xE_x}$ can be calculated explicitly. This is indeed possible for a uniform charge distribution within an elliptical boundary, in which case E_x is a linear function of x (see section 3).

Hence we obtain the r.m.s. envelope equation

$$\frac{d^2}{ds^2}\tilde{x} + k_x(s)\tilde{x} - \frac{\varepsilon_x^2}{16\tilde{x}^3} - \frac{qI}{4\pi\varepsilon_0 m\gamma^3 v_0^2(\tilde{x}+\tilde{z})} = 0 \tag{A.11}$$

and similar in z. Note that $\tilde{x} = a/2$ and $\tilde{z} = b/2$, where a, b are the usual envelopes.

We also derive from Eq. (A.10) the differential equation

$$\frac{d}{ds}\varepsilon_x^2 = \frac{32q}{m\gamma^3 v_0^2}\left(\overline{x^2}\;\overline{x'E_x} \;-\; \overline{xx'}\;\overline{xE_x}\right) \tag{A.12}$$

and similar for z, where we had to introduce the moments $\overline{xE_x}, \overline{x'E_x}$. For E_x a linear function of x, one readily sees that ε_x must be constant, hence the r.m.s. envelope equations are a closed set of ordinary differential equations and can be solved explicitly. For E_x other than a linear function of x (or a constant) these moments are of higher order, hence the above differential equations are not a closed set. By going to higher order moments we would obtain an infinite set of coupled equations, in general.

A.2 Generalized Emittance Equation

In the following we show that it is possible to transform the terms involving the electric field in such a way that only the energy of the field appears explicitly. The latter still includes higher-order moments, but we benefit from the fact that it is a quantity of direct physical meaning and amenable to estimates.

For this purpose we re-write

$$\overline{x'E_x} = N^{-1}\int ... \int x'E_x f \; dx...dz' = N^{-1}\int\int E_x n v_x \; dx \; dz \tag{A.13}$$

where we have introduced v_x as local averaged velocity of beam particles (in a frame, where the beam is at rest).

With the local current

$$\mathrm{j} = qnv_0\mathrm{v} \tag{A.14}$$

we obtain

$$\overline{x'E_x} = (Nqv_0)^{-1}\int\int E_x j_x dx \; dz \tag{A.15}$$

and similar for z. We thus find, using $\mathbf{E} = -\nabla\phi$):

$$\int\int \mathbf{E}\cdot\mathbf{j}\ dx\ dz = \int\int \phi\nabla\cdot\mathbf{j}\ dx\ dz = -qv_0\int\int \phi\frac{\delta n}{\delta s}\ dx\ dz \tag{A.16}$$

where we have used the continuity equation

$$\frac{\delta n}{\delta s} + (qv_0)^{-1}\nabla\cdot\mathbf{j} = 0 \tag{A.17}$$

derived from Eq. (A.4) by integration. The integration in Eq. (A.16) is performed over the cross section F, which contains the beam in its interior, hence we may neglect a boundary integral. Using Poisson's equation we obtain by partial integration:

$$\int\int \mathbf{E}\cdot\mathbf{j}\ dx\ dz = -v_0\frac{d}{ds}W - \varepsilon_0 v_0\int \phi\frac{\partial}{\partial s}E_n d\sigma \tag{A.18}$$

with E_n the normal component of $\mathbf{E}$ on the boundary of F (surface element $d\sigma$) and

$$W = \frac{\varepsilon_0}{2}\int\int \mathbf{E}^2 dx\ dz \tag{A.19}$$

the electric field energy within F.

The next step is to add the three Eqs. (A.12) and express $\overline{x'E_x}$ by the electric field energy according to Eqs. (A.15, A.18). We thus find the relationship

$$\frac{1}{\overline{\overline{x^2}}}\frac{d}{ds}\varepsilon_x^2 + \frac{1}{\overline{\overline{z^2}}}\frac{d}{ds}\varepsilon_z^2 = \frac{32q}{m\gamma^3 v_0^2}\left[-\frac{1}{Nq}\frac{dW}{ds} - \frac{\varepsilon_0}{Nq}\int \phi\frac{\delta}{\delta s}E_n d\sigma - I\right] \tag{A.20}$$

with

$$I = \frac{1}{2}\left(\frac{1}{\overline{\overline{x^2}}}\frac{d\overline{x^2}}{ds}\overline{xE_x} + \frac{1}{\overline{\overline{z^2}}}\frac{d\overline{z^2}}{ds}\ \overline{zE_z}\right). \tag{A.21}$$

Equation (A.20) holds exactly, but we need some approximation to evaluate the term I on the r.h.s. and obtain a practically useful equation.

Firstly, it can be shown that for a uniform beam with elliptic cross section the field energy per unit length calculated within a large circle of radius R is given by

$$W_u = \frac{N^2q^2}{16\pi\varepsilon_0}\left(1 + 4\ ln\frac{2R}{a+b}\right) \tag{A.22}$$

where a and b are the semi-axi in x and z. We then also find for the uniform beam that I is related to W_u according to

$$I = -\frac{1}{Nq}\frac{d}{ds}W_u \tag{A.23}$$

Next we use the result found by Sacherer that $\overline{xE_x}$ and $\overline{zE_z}$ are independent of the density profile as long as elliptical symmetry is given:

$$n(x,z,s) = n\left(\frac{x^2}{a^2} + \frac{z^2}{b^2}, s\right) \tag{A.24}$$

Hence, Eq. (A.23) is true for all beams satisfying equation (A.24) and we re-write Eq. (A.20) as

$$\frac{1}{\overline{\overline{x^2}}}\frac{d}{ds}\varepsilon_x^2 + \frac{1}{\overline{\overline{z^2}}}\frac{d}{ds}\varepsilon_z^2 = -4K\frac{d}{ds}\frac{W - W_u}{w_0}. \tag{A.25}$$

Here we have neglected the boundary integral (which is justified for large R) and introduced the generalized perveance

$$K \equiv \frac{Nq^2}{2\pi\varepsilon_0 m\gamma^3 v_0^2} \tag{A.26}$$

and the field energy normalization constant $w_0 \equiv (N^2q^2)/(16\pi\varepsilon_0)$, which gives the field energy of a uniform beam within the actual beam boundary. We note that the generalized emittance Eq. (A.25) derived here for 2-D beams can be derived for 3-D bunched beams as well [7].